

Galois correspondence for counting quantifiers

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We introduce a new type of closure operator on the set of relations, max-implementation, and its weaker analog max-quantification. Then we show that approximation preserving reductions between counting constraint satisfaction problems (#CSPs) are preserved by these two types of closure operators. Together with some previous results this means that the approximation complexity of counting CSPs is determined by partial clones of relations that additionally closed under these new types of closure operators. Galois correspondence of various kind have proved to be quite helpful in the study of the complexity of the CSP. While we were unable to identify a Galois correspondence for partial clones closed under max-implementation and max-quantification, we obtain such results for slightly different type of closure operators, k -existential quantification. This type of quantifiers are known as counting quantifiers in model theory, and often used to enhance first order logic languages. We characterize partial clones of relations closed under k -existential quantification as sets of relations invariant under a set of partial functions that satisfy the condition of k -subset surjectivity. Finally, we give a description of Boolean max-co-clones, that is, sets of relations on $\{0, 1\}$ closed under max-implementations.

This is an extended version of [12].

Key words: counting constraint satisfaction problem, approximation, co-clones, Galois correspondence

1 INTRODUCTION

Clones of functions and clones of relations in their various incarnations have proved to be an immensely powerful tool in the study of the complexity of different versions of the Constraint Satisfaction Problem (CSP, for short). In a CSP the aim is to find an assignment of values to a given set of variables, subject to constraints on the values that can be assigned simultaneously to certain specified subsets of variables. A CSP can also be expressed as the problem of deciding whether a given conjunctive formula has a model. In the counting version of the CSP the goal is to find the number of satisfying assignments, and in the quantified version we need to verify if a first order sentence, whose quantifier-free part is conjunctive, is true in a given model.

The general CSP is NP-complete [26]. However, many practical and theoretical problems can be expressed in terms of CSPs using constraints of a certain restricted form. One of the most widely used way to restrict a constraint satisfaction problem is to specify the set of allowed constraints, which is usually a collection of relations on a finite set. The key result is that this set of relations can usually be assumed to be a co-clone of a certain kind. More precisely, a generic statement asserts that if a relation R belongs to the co-clone generated by a set Γ of relations then the CSP over $\Gamma \cup \{R\}$ is polynomial time reducible to the CSP over Γ . Then we can use the appropriate Galois connection to transfer the question about sets of relations to a question about certain classes of functions.

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For the classical decision CSP such a result was obtained by Jeavons et al. [25], who proved that intersection of relations (that is, conjunction of the corresponding predicates) and projections (that is, existential quantification) give rise to polynomial time reducibility of CSPs. Therefore in the study of the complexity of the CSP it suffices to focus on co-clones. Using the result of Geiger [21] or the one of Bodnarchuk et al. [3] one can instead consider clones of functions. A similar result is true for the counting CSP as shown by Bulatov and Dalmau [9]. In the case of quantified CSP, Börner et al. proved [4] that conjunction, existential quantification, and also universal quantification give rise to a polynomial time reduction between quantified problems. The appropriate class of functions is then the class of surjective functions. Along with the usual counting CSP, a version, in which one is required to approximate the number of solutions, has also been considered. The standard polynomial time reduction between problems is not suitable for approximation complexity. In this case, therefore, another type of reductions, approximation preserving, or, AP-reductions, is used. The first author proved in [8] that conjunction of predicates gives rise to an AP-reduction between approximation counting CSPs. By the Galois connection established by Fleischer and Rosenberg [20], the approximation complexity of a counting CSP is a property of a clone of partial functions.

In most cases establishing the connection between clones of functions and reductions between CSPs has led to a major success in the study of the CSP. For the decision problem, a number of very strong results have been proved using methods of universal algebra [10, 5, 6, 2, 23]. For the exact counting CSP a complete complexity classification of such problems has been obtained [7]. Substantial progress has been also made in the case of quantified CSP [13].

Compared to the results cited above the progress made in the approximation counting CSP is modest. Perhaps, one reason for this is that clones of partial functions are much less studied, and much more diverse than clones of total functions. In this paper we attempt to overcome to some extent the difficulties arising from this weakness of partial clones.

In the first part of the paper we introduce new types of quantification and show that such quantifications, we call them max-implementation and max-quantification, give rise to AP-reductions between approximation counting CSPs. Intuitively, applying the max-quantifier to a relation $R(x_1, \dots, x_n, y)$ results in the relation $\exists_{\max}^1 y R(x_1, \dots, x_n, y)$ that contains those tuples (a_1, \dots, a_n) that have a maximal number of extensions (a_1, \dots, a_n, b) such that $R(a_1, \dots, a_n, b)$ is satisfied. Max-implementation, \exists_{\max} , is a similar construction, but applied to a group of variables. Sets of relations closed with respect this new type of quantification will be called max-co-clones. Thus we strengthen the closure operator on sets of relation hoping that the sets of functions corresponding to the new type of Galois connection are easier to study. We were unable, however, to describe a Galois connection for sets closed under max-implementation and max-quantification. Instead, we consider a somewhat close type of quantifiers, k -existential quantifiers. Quantifiers of this type are known as counting quantifiers in model theory, and often used to enhance first order logic languages (see, e.g. [16]). Counting quantifiers are similar to max-existential quantifiers, although do not capture them completely. We call sets of relations closed under conjunctions and k -existential quantification k -existential co-clones. On the functional side, an n -ary (partial) function on a set D is said to be k -subset surjective if it is surjective on any collection of k -element subsets. More precisely, for any k -element subsets $A_1, \dots, A_n \subseteq D$ the set $f(A_1, \dots, A_n)$ contains at least k elements. The second result of the paper asserts that k -existential co-clones are exactly the sets of relation invariant with respect to a set of k -subset surjective (partial) functions. Finally, we give a complete description of max-co-clones on $\{0, 1\}$ (Boolean max-co-clones). Surprisingly, any Boolean max-co-clone is also a usual co-clone (but not the other way around). We show that in general it is not true.

2 PRELIMINARIES

By $[n]$ we denote the set $\{1, \dots, n\}$. For a set D , by D^n we denote the set of all n -tuples of elements of D . An n -ary relation is any set $R \subseteq D^n$. The number n is called the *arity* of R and denoted $\text{ar}(R)$. Tuples will be denoted in boldface, say, \mathbf{a} , and their entries will be denoted by $\mathbf{a}[1], \dots, \mathbf{a}[n]$. For $I = (i_1, \dots, i_k) \subseteq [n]$ by $\text{pr}_I \mathbf{a}$ we denote the tuple $(\mathbf{a}[i_1], \dots, \mathbf{a}[i_k])$, and we use $\text{pr}_I R$ to denote $\{\text{pr}_I \mathbf{a} \mid \mathbf{a} \in R\}$. We will also need predicates corresponding to relations. To simplify the notation we use the same symbol for a relation and the corresponding predicate, for instance,

for an n -ary relation R the corresponding predicate $R(x_1, \dots, x_n)$ is given by $R(\mathbf{a}[1], \dots, \mathbf{a}[n]) = 1$ if and only if $\mathbf{a} \in R$. Relations and predicates are used interchangeably.

For a set of relations Γ over a set D , the set $\langle\langle\Gamma\rangle\rangle$ includes all relations that can be expressed (as a predicate) using (a) relations from Γ , together with the binary equality relation $=_D$ on D , (b) conjunctions, and (c) existential quantification. This set is called the *co-clone generated by Γ* .

Partial co-clone generated by Γ is obtained in a similar way by disallowing existential quantification. $\langle\Gamma\rangle$ includes all relations that can be expressed using (a) relations from Γ , together with $=_D$, and (b) conjunctions,

If $\Gamma = \langle\Gamma\rangle$ or $\Gamma = \langle\langle\Gamma\rangle\rangle$, the set Γ is said to be a *partial co-clone*, and a *co-clone*, respectively.

Sometimes there is no need to apply even conjunction to produce a new relation. For instance, $Q(x, y) = R(x, y, y)$ defines a binary relation from a ternary one. Therefore it is often convenient, especially for technical purposes, to group manipulations with variables of a relation into a separate category. More formally, for a relation $R(x_1, \dots, x_n)$ and a mapping $\pi: \{x_1, \dots, x_n\} \rightarrow V$, where V is some set of variables, πR denotes the relation $R(\pi(x_1), \dots, \pi(x_n))$. We will understand by (partial) co-clones sets of relations closed under manipulation with variables, conjunction, and existential quantification (respectively, closed under manipulation with variables and conjunction).

Co-clones and partial co-clones can often be conveniently and concisely represented through functions and partial functions, respectively.

Let R be a (k -ary) relation on a set D , and $f: D^n \rightarrow D$ an n -ary function on the same set. Function f *preserves* R , or is a *polymorphism* of R , if for any n tuples $\mathbf{a}_1, \dots, \mathbf{a}_n \in R$ the tuple $f(\mathbf{a}_1, \dots, \mathbf{a}_n)$ obtained by component-wise application of f also belongs to R . Relation R in this case is said to be *invariant* with respect to f . The set of all functions that preserve every relation from a set of relations Γ is denoted by $\text{Pol}(\Gamma)$, the set of all relations invariant with respect to a set of functions C is denoted by $\text{Inv}(C)$.

Operators Inv and Pol form a Galois connection between sets of functions and sets of relations. Sets of the form $\text{Inv}(C)$ are precisely co-clones; on the functional side there is another type of closed sets.

A set of functions is said to be a *clone* of functions if it is closed under superpositions and contain all the *projection* functions, that is functions of the form $f(x_1, \dots, x_n) = x_i$. Sets of functions of the form $\text{Pol}(\Gamma)$ are exactly clones of functions [27].

The study of the #CSP also makes use of another Galois connection, a connection between partial co-clones and sets of *partial functions*. An n -ary partial function f on a set D is just a partial mapping $f: D^n \rightarrow D$. As in the case of total functions, a partial function f *preserves* relation R , if for any n tuples $\mathbf{a}_1, \dots, \mathbf{a}_n \in R$ the tuple $f(\mathbf{a}_1, \dots, \mathbf{a}_n)$ obtained by component-wise application of f is either undefined or belongs to R . The set of all partial functions that preserve every relation from a set of relations Γ is denoted by $\text{pPol}(\Gamma)$.

The set of all tuples from D^n on which f is defined is called the *domain* of f and denoted by $\text{Dom}(f)$. A set of functions is said to be *down-closed* if along with a function f it contains any function f' such that $\text{Dom}(f') \subseteq \text{Dom}(f)$ and $f'(a_1, \dots, a_n) = f(a_1, \dots, a_n)$ for every tuple $(a_1, \dots, a_n) \in \text{Dom}(f')$. A down-closed set of functions, containing all projections and closed under superpositions is called a *partial clone*. Fleischer and Rosenberg [20] proved that partial clones are exactly the sets of the form $\text{pPol}(\Gamma)$ for a certain Γ , and that the partial co-clones are precisely the sets $\text{Inv}(C)$ for collections C of partial functions.

3 APPROXIMATE COUNTING AND MAX-IMPLEMENTATION

Let D be a set, and let Γ be a finite set of relations over D . An instance of the counting Constraint Satisfaction Problem, $\#\text{CSP}(\Gamma)$, is a pair $\mathcal{P} = (V, \mathcal{C})$ where V is a set of *variables*, and \mathcal{C} is a set of *constraints*. Every constraint is a pair $\langle \mathbf{s}, R \rangle$, in which R is a member of Γ , and \mathbf{s} is a tuple of variables from V of length $\text{ar}(R)$ (possibly with repetitions). A *solution* to \mathcal{P} is a mapping $\varphi: V \rightarrow D$ such that $\varphi(\mathbf{s}) \in R$ for every constraint $\langle \mathbf{s}, R \rangle \in \mathcal{C}$. The objective in $\#\text{CSP}(\Gamma)$ is to find the number $\#\mathcal{P}$ of solutions to a given instance \mathcal{P} .

We are interested in the complexity of this problem depending on the set Γ . The complexity of the exact counting problem (when we are required to find the exact number of solutions) is settled in [7] by showing that for any finite D

and any set Γ of relations over D the problem is polynomial time solvable or is complete in a natural complexity class $\#P$. One of the key steps in that line of research is the following result: For a relation R and a set of relations Γ over D , if R belongs to the co-clone generated by Γ , then $\#CSP(\Gamma \cup \{R\})$ is polynomial time reducible to $\#CSP(\Gamma)$. This results emphasizes the importance of co-clones in the study of constraint problems.

A situation is different when we are concerned about approximating the number of solutions. We will need some notation and terminology. Let A be a counting problem. An algorithm Alg is said to be an *approximation algorithm* for A with relative error ε (which may depend on the size of the input) if it is polynomial time and for any instance \mathcal{P} of A it outputs a certain number $\text{Alg}(\mathcal{P})$ such that $\text{Alg}(\mathcal{P}) = 0$ if \mathcal{P} has no solution and

$$\frac{|\#\mathcal{P} - \text{Alg}(\mathcal{P})|}{\#\mathcal{P}} < \varepsilon$$

otherwise, where $\#\mathcal{P}$ denotes the exact number of solutions to \mathcal{P} .

The following framework is viewed as one of the most realistic models of efficient computations. A *fully polynomial approximation scheme* (FPAS, for short) for a problem A is an algorithm Alg such that: It takes as input an instance \mathcal{P} of A and a real number $\varepsilon > 0$, the relative error of Alg on the input $(\mathcal{P}, \varepsilon)$ is less than ε , and Alg is polynomial time in the size of \mathcal{P} and $\log(\frac{1}{\varepsilon})$.

To determine the approximation complexity of problems approximation preserving of reductions are used. Suppose A and B are two counting problems whose complexity (of approximation) we want to compare. An *approximation preserving reduction* or *AP-reduction* from A to B is an algorithm Alg , using B as an oracle, that takes as input a pair $(\mathcal{P}, \varepsilon)$ where \mathcal{P} is an instance of A and $0 < \varepsilon < 1$, and satisfies the following three conditions: (i) every oracle call made by Alg is of the form (\mathcal{P}', δ) , where \mathcal{P}' is an instance of B , and $0 < \delta < 1$ is an error bound such that $\log(\frac{1}{\delta})$ is bounded by a polynomial in the size of \mathcal{P} and $\log(\frac{1}{\varepsilon})$; (ii) the algorithm Alg meets the specifications for being an FPAS for A whenever the oracle meets the specification for being an FPAS for B ; and (iii) the running time of Alg is polynomial in the size of \mathcal{P} and $\log(\frac{1}{\varepsilon})$. If an approximation preserving reduction from A to B exists we denote it by $A \leq_{\text{AP}} B$, and say that A is *AP-reducible* to B .

Similar to co-clones and polynomial time reductions, partial co-clones can be shown to be preserved by AP-reductions.

Theorem 1 ([8]) *Let R be a relation and Γ be a set of relations over a finite set such that R belongs to $\langle \Gamma \rangle$. Then $\#CSP(\Gamma \cup \{R\})$ is AP-reducible to $\#CSP(\Gamma)$.*

This result however has two significant setbacks. First, partial co-clones are not studied to the same extent as regular co-clones, and, due to greater diversity, are not believed to be ever studied to a comparable level. Second, it does not use the full power of AP-reductions, and therefore leaves significant space for improvements. In the rest of this section we try to improve upon the second issue.

Definition 2 *Let Γ be a set of relations on a set D , and let R be an n -ary relation on D . Let \mathcal{P} be an instance of $\#CSP(\Gamma)$ over the set of variables consisting of $V = V_x \cup V_y$, where $V_x = \{x_1, x_2, \dots, x_n\}$ and $V_y = \{y_1, y_2, \dots, y_q\}$. For any assignment of $\varphi : V_x \rightarrow D$, let $\#\varphi$ be the number of assignments $\psi : V_y \rightarrow D$ such that $\varphi \cup \psi$ satisfy \mathcal{P} . Let M be the maximum value of $\#\varphi$ among all assignments of V_x . The instance \mathcal{P} is said to be a max-implementation of R if a tuple φ is in R if and only if $\#\varphi = M$.*

Theorem 3 *If there is max-implementation of R by Γ , then $\#CSP(\Gamma \cup \{R\}) \leq_{\text{AP}} \#CSP(\Gamma)$.*

Proof: Let $\mathcal{P} = (V = V_x \cup V_y, \mathcal{C})$ be a max-implementation of R by Γ , and let M be the maximal number of extensions of assignments of V_x to solutions of \mathcal{P} . For any instance $\mathcal{P}_1 = (V_1, \mathcal{C}_1)$ of $\#CSP(\Gamma \cup \{R\})$ we construct an instance $\mathcal{P}_2 = (V_2, \mathcal{C}_2)$ of $\#CSP(\Gamma)$ as follows.

- Choose a sufficiently large integer m (to be determined later).

- Let $C_1, \dots, C_\ell \in \mathcal{C}_1$ be the constraints from \mathcal{P}_1 involving R , $C_i = \langle s_i, R \rangle$. Set $V_2 = V_1 \cup \bigcup_{i=1}^\ell (V_1^i \cup \dots \cup V_m^i)$, where each V_j^i is a fresh copy of V_y .
- Let \mathcal{C} be the set of constraints of \mathcal{P} . Set $\mathcal{C}_2 = (\mathcal{C}_1 - \{C_1, \dots, C_\ell\}) \cup \bigcup_{i=1}^\ell (\mathcal{C}_1^i \cup \dots \cup \mathcal{C}_m^i)$, where each \mathcal{C}_j^i is a copy of \mathcal{C} defined as follows. For each $\langle s, Q \rangle \in \mathcal{C}$ we include $\langle s_j^i, Q \rangle$ into \mathcal{C}_j^i , where s_j^i is obtained from s replacing every variable from V_y with its copy from V_j^i .

Now, as is easily seen, every solution of \mathcal{P}_1 can be extended to a solution of \mathcal{P}_2 in $M^{\ell m}$ ways. Observe that sometimes the restriction of a solution ψ of \mathcal{P}_2 to V_1 is not a solution of \mathcal{P}_1 . Indeed, it may happen that although ψ satisfies every copy \mathcal{C}_j^i of \mathcal{P} , its restriction to s_j^i does not belong to R , simply because this restriction does not have sufficiently many extensions to solutions of \mathcal{P} . However, any assignment to V_1 that is not a solution to \mathcal{P}_1 can be extended to a solution of \mathcal{P}_2 in at most $(M-1)^m \cdot M^{(\ell-1)m}$ ways. Hence,

$$M^{\ell m} \cdot \#\mathcal{P}_1 \leq \#\mathcal{P}_2 \leq M^{\ell m} \cdot \#\mathcal{P}_1 + |V_1|^{|D|} \cdot (M-1)^m \cdot M^{(\ell-1)m}$$

Then we output $\frac{\#\mathcal{P}_2}{M^{\ell m}}$.

Let $|V_1| = k$ and $|D| = d$. Given a desired relative error ε we have to find m such that

$$\frac{\frac{\#\mathcal{P}_2}{M^{\ell m}} - \#\mathcal{P}_1}{\#\mathcal{P}_1} < \varepsilon.$$

A straightforward computation shows that any

$$m > \frac{d \log k - \log \varepsilon}{\log(M-1) - \log M}$$

achieves the goal. \square

Max-implementation can be used as another closure operator on the set of relations. Let $R(x_1, \dots, x_n, y_1, \dots, y_m)$ be a relation on a set D . By $\exists_{\max}(y_1, \dots, y_m)R(x_1, \dots, x_n, y_1, \dots, y_m)$ we denote the relation $Q(x_1, \dots, x_n)$ on the same set given by the rule: $\mathbf{a} \in Q$ if and only if there are M tuples $\mathbf{b} \in D^m$ such that $(\mathbf{a}, \mathbf{b}) \in R$, where M is the maximal number of elements in the set $\{\mathbf{b} \mid (\mathbf{a}, \mathbf{b}) \in Q\}$ over all $\mathbf{a} \in D^n$. A set of relations Γ over D is said to be a *max-co-clone* if it contains the equality relations, and closed under conjunctions and max-implementations. The smallest max-co-clone containing a set of relations Γ is called the *max-co-clone generated by Γ* and denoted $\langle \Gamma \rangle_{\max}$.

Lemma 4 *Let Γ be a set of relations and $R \in \langle \Gamma \rangle_{\max}$. Then there is a max-implementation of R by Γ .*

Proof: Suppose $R \in \langle \Gamma \rangle_{\max}$. We need to show that R can be represented as $R(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m) \Phi(x_1, \dots, x_n, y_1, \dots, y_m)$, where Φ is quantifier free. To this end it suffices to prove three equalities:

1. if $R(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m) \Phi(x_1, \dots, x_n, y_1, \dots, y_m)$ and π is a transformation of the set $\{x_1, \dots, x_n\}$ then $(\pi R)(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m) \Phi(\pi(x_1), \dots, \pi(x_n), y_1, \dots, y_m)$;
2. if $R(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m) \Phi_1(x_1, \dots, x_n, y_1, \dots, y_m) \wedge \exists_{\max}(z_1, \dots, z_r) \Phi_2(x_1, \dots, x_n, z_1, \dots, z_r)$, then $R(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m, z_1, \dots, z_r) (\Phi_1(x_1, \dots, x_n, y_1, \dots, y_m) \wedge \Phi_2(x_1, \dots, x_n, z_1, \dots, z_r))$;
3. if $R(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m) \exists_{\max}(z_1, \dots, z_r) \Phi(x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_r)$, then there is a quantifier free formula Ψ such that $R(x_1, \dots, x_n) = \exists_{\max}(u_1, \dots, u_s) \Psi(x_1, \dots, x_n, u_1, \dots, u_s)$.

(1) follows straightforwardly from definitions.

(2) $\mathbf{a} \in R$ if and only if it has the maximal number of extensions in both Φ_1 and Φ_2 . Without loss of generality, sets $\{y_1, \dots, y_m\}$ and $\{z_1, \dots, z_r\}$ are disjoint. Let a tuple $\mathbf{a} \in R$ have M_1 extensions in Φ_1 and M_2 extensions in

Φ_2 . Then it has $M_1 M_2$ extensions in $\Phi_1 \wedge \Phi_2$. On the other hand, let $\mathbf{a} \notin R$. Let also it have M'_1 extensions in Φ_1 and M'_2 extensions in Φ_2 , and either $M'_1 < M_1$ or $M'_2 < M_2$. Since such tuple has $M'_1 M'_2 < M_1 M_2$ extensions, it does not belong to the relation defined by $R(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m, z_1, \dots, z_r)(\Phi_1(x_1, \dots, x_n, y_1, \dots, y_m) \wedge \Phi_2(x_1, \dots, x_n, z_1, \dots, z_r))$ as well.

(3) Observe first that $R(x_1, \dots, x_n)$ does not necessarily equal $\exists_{\max}(y_1, \dots, y_m, z_1, \dots, z_r)\Phi(x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_r)$. Indeed, let Φ' denote the formula $Q(x_1, \dots, x_n, y_1, \dots, y_m) = \exists(z_1, \dots, z_r)\Phi(x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_r)$. Then it is possible that although every extension of a tuple \mathbf{a} to $(\mathbf{a}, \mathbf{b}) \in Q$ has very few extensions to a tuple from Φ , and so $\mathbf{a} \notin R$, the number of extensions \mathbf{b} is large so that combined \mathbf{a} has enough extensions to tuples from Φ .

To avoid this we make sure that extensions to tuples from Q cannot make up for extensions to Φ . Let M be the maximal number of extensions \mathbf{b} of tuple \mathbf{a} such that $(\mathbf{a}, \mathbf{b}) \in Q$, and N the maximal number of extensions \mathbf{c} of $(\mathbf{a}, \mathbf{b}) \in Q$ to $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Phi$. Let also L be the maximal number of extensions \mathbf{b} of $\mathbf{a} \in R$; it is possible that $L < M$. Set

$$c = \max\left(1, \left\lceil \log \frac{L}{M} / \log \frac{N-1}{N} \right\rceil\right).$$

We show that $R(x_1, \dots, x_n) = \exists_{\max}(u_1, \dots, u_s)\Psi(x_1, \dots, x_n, u_1, \dots, u_s)$, where $\{u_1, \dots, u_s\} = \{y_1, \dots, y_m, z_1^1, \dots, z_r^1, \dots, z_1^c, \dots, z_r^c\}$, and

$$\Psi(x_1, \dots, x_n, u_1, \dots, u_s) = \bigwedge_{s=1}^c \Phi(x_1, \dots, x_n, y_1, \dots, y_m, z_1^s, \dots, z_r^s).$$

If a tuple \mathbf{a} belongs to R it is extendable in L ways to a tuple from Q , and then every such extended tuple (\mathbf{a}, \mathbf{b}) is extendable in N ways to a tuple from Φ . Therefore \mathbf{a} has LN^c extensions to a tuple from Ψ . On the other hand, if $\mathbf{a} \notin R$, then it can be extended in at most M ways to a tuple $(\mathbf{a}, \mathbf{b}) \in Q$, then this tuple is extendable in at most $N-1$ ways to a tuple from Φ . Thus $\mathbf{a} \notin R$ has

$$M(N-1)^c = LN^c \cdot \frac{M}{L} \left(\frac{N-1}{N}\right)^c < LN^c$$

extensions. □

The next natural step would be to find a type of functions and a closure operator on the set of functions that give rise to a Galois connection capturing max-co-clones.

Problem 1 Find a class \mathcal{F} of (partial) functions and a closure operator $[\cdot]$ on this class such that for any set of relations Γ and any set $C \subseteq \mathcal{F}$ it holds that $\langle \Gamma \rangle_{\max} = \text{Inv}(\mathcal{F} \cap \text{pPol}(\Gamma))$, and $[C] = \mathcal{F} \cap \text{pPol}(\text{Inv}(C))$.

In all the cases previously studied the projection (or quantification) type operators on relations can be reduced to quantifying away a single variable. However, max-implementations seem to inherently involve a number of variables, rather than a single variable. In the end of this paper we use our description of Boolean max-co-clones to show that max-implementations are provably more powerful than max-quantification (see below). In the Boolean case every max-quantification is equivalent to either existential quantification, or universal quantification. Sets of relations on $\{0, 1\}$ closed under these two types of quantifications are well known: these are sets of invariant relations of sets of surjective functions [4]. However, not all of them are max-co-clones.

Therefore a meaningful relaxation of max-co-clones restricts the use of max-implementation to one auxiliary variable. Let Φ be a formula with free variables x_1, \dots, x_n and y over set D and some predicate symbols. Then a_1, \dots, a_n satisfy

$$\Psi(x_1, \dots, x_n) = \exists_{\max}^1 y \Phi(x_1, \dots, x_n, y)$$

if and only if the number of $b \in D$ such that $\Phi(a_1, \dots, a_n, b)$ is true is maximal among all tuples $(c_1, \dots, c_n) \in D^n$. The quantifier \exists_{\max}^1 will be called *max-quantifier*. A set of relations Γ over D is said to be a *max-existential co-clone* if it contains the equality relation, and closed under conjunctions and max-existential quantification. The smallest max-existential co-clone containing a set of relations Γ is called the *max-existential co-clone generated by Γ* and denoted $\langle \Gamma \rangle_{\max}^1$.

Problem 2 Find a class \mathcal{F} of (partial) functions and a closure operator $[\cdot]$ on this class such that for any set of relations Γ and any set of functions $C \subseteq \mathcal{F}$ it holds that $\langle \Gamma \rangle_{\max}^1 = \text{Inv}(\mathcal{F} \cap \text{pPol}(\Gamma))$, and $[C] = \mathcal{F} \cap \text{pPol} \text{Inv}(C)$.

In the next section we consider certain constructions approximating max-existential co-clones.

4 K-EXISTENTIAL AND MAX-EXISTENTIAL CO-CLONES

In order to approach max-quantification we consider counting quantifiers that have been used in model theory to increase the power of first order logic [24, 19].

Let Φ be a formula with free variables x_1, \dots, x_n and y over set D and some predicate symbols. Then a_1, \dots, a_n satisfy

$$\Psi(x_1, \dots, x_n) = \exists_k y \Phi(x_1, \dots, x_n, y)$$

if and only if $\Phi(a_1, \dots, a_n, b)$ is true for at least k values $b \in D$. The quantifier \exists_k will be called *k-existential quantifier*. It is easy to see that 1-existential quantifier is just the regular existential quantifier, and the $|D|$ -existential quantifier is equivalent to the universal quantifier on set D .

We now introduce several types of co-clones depending on what kind of k -existential quantifiers are allowed. A set of relations Γ over set D is said to be a *k-existential partial co-clone* if it contains the equality relation $=_D$, and closed under manipulations with variables, conjunction, and k -existential quantification. The smallest k -existential partial co-clone containing a set of relations Γ is called the *k-existential partial co-clone generated by Γ* and denoted $\langle \Gamma \rangle_k$. In a similar way we can define sets of relations closed under several counting quantifiers. Let $K \subseteq \mathbb{N}$. A set of relations Γ over set D is said to be a *K-existential partial co-clone* if it contains the equality relation $=_D$, and closed under manipulations with variables, conjunction, and k -existential quantification for $k \in K$. Clearly, if Γ is a set of relations on an m -element set, we may assume $K \subseteq [m]$. If $1 \in K$, set Γ is closed under existential quantification, and so it is called a *K-existential co-clone*. If, in addition, $K = \{1, k\}$, Γ is called *k-existential co-clone*. The set Γ is said to be a *counting co-clone** if it is an \mathbb{N} -existential partial co-clone, that is, if it contains $=_D$, and closed under conjunctions and k -existential quantification for all $k \geq 1$. The smallest K -existential partial co-clone (K -existential co-clone, k -existential co-clone, counting co-clone) containing Γ are called the *K-existential partial co-clone* (*K-existential co-clone*, *k-existential co-clone*, *counting co-clone*) generated by Γ and denoted $\langle \Gamma \rangle_K$ ($\langle \langle \Gamma \rangle \rangle_K$, $\langle \langle \Gamma \rangle \rangle_k$, $\langle \langle \Gamma \rangle \rangle_\infty$, respectively).

We observe some simple properties of counting quantifiers.

Lemma 5 Let $\Phi(x_1, \dots, x_n, y_1, \dots, y_m)$ and $\Psi(x_1, \dots, x_n, z_1, \dots, z_\ell)$ be conjunctive quantifier free formulas. Then

$$\begin{aligned} & \exists_{s_1} y_1 \dots \exists_{s_m} y_m \exists_{t_1} z_1 \dots \exists_{t_\ell} (\Phi(x_1, \dots, x_n, y_1, \dots, y_m) \wedge \Psi(x_1, \dots, x_n, z_1, \dots, z_\ell)) \\ &= (\exists_{s_1} y_1 \dots \exists_{s_m} y_m (\Phi(x_1, \dots, x_n, y_1, \dots, y_m))) \wedge (\exists_{t_1} z_1 \dots \exists_{t_\ell} \Psi(x_1, \dots, x_n, z_1, \dots, z_\ell)), \end{aligned}$$

for any $s_1, \dots, s_m, t_1, \dots, t_\ell \in \mathbb{N}$, provided $y_1, \dots, y_m, z_1, \dots, z_\ell \notin \{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\} \cap \{z_1, \dots, z_\ell\} = \emptyset$.

Corollary 6 Let Γ be a set of relations on a set D , $K \subseteq \mathbb{N}$, and $R(x_1, \dots, x_n) \in \langle \Gamma \rangle_K$. Then there is a conjunctive quantifier free formula $\Phi(x_1, \dots, x_n, y_1, \dots, y_m)$ using relations from Γ and the equality relation such that

$$R(x_1, \dots, x_n) = \exists_{s_1} \dots \exists_{s_m} \Phi(x_1, \dots, x_n, y_1, \dots, y_m).$$

* ‘Counting’ in this term comes from counting quantifiers and has nothing to do with counting constraint satisfaction.

The following observation summarizes some relationship between the constructions introduced.

Observation 7 *For a set of relations Γ on D , $|D| = m$, the following hold.*

- Γ is a 1-existential (partial) co-clone if and only if it is a co-clone.
- Γ is a (partial) m -existential clone if and only if it is a (partial) co-clone closed under universal quantification.
- if Γ is a counting co-clone then it is a max-existential co-clone.
- if Γ is a max-existential co-clone then it is a partial m -existential co-clone.

In all other cases the introduced versions of co-clones are incomparable.

Example 8 Fix a natural number m and let D be a set with $\frac{m(m-1)}{2}$ elements. Consider an equivalence relation R_m on D with classes D_1, \dots, D_m such that $|D_i| = i$. Then the co-clone generated by R_m corresponds to one of the Rosenberg's maximal clones [29], and so the structure of relations from this co-clone is well understood. For any n -ary relation $Q \in \langle\langle R_m \rangle\rangle$ there is a partition I_1, \dots, I_k of $[n]$ such that a tuple \mathbf{a} belongs to Q if and only if for each $j \in [k]$ and every $i, i' \in I_j$ the entries $\mathbf{a}[i], \mathbf{a}[i']$ are R_m -related. This also means that $\langle R_m \rangle = \langle\langle R_m \rangle\rangle$.

Applying k -existential and max-existential quantifiers one can easily find the k -existential, counting, and max-existential clones generated by R :

1. $\langle R_m \rangle_k = \langle\langle R_m \rangle\rangle_k$ is the set of relations Q : There is a partition I_1, \dots, I_t of $[\text{ar}(Q)]$ and $J \subseteq [t]$ such that a tuple \mathbf{a} belongs to Q if and only if for each $j \in [t]$ and every $i, i' \in I_j$ the entries $\mathbf{a}[i], \mathbf{a}[i']$ are R_m -related and $\mathbf{a}[i] \in D_k \cup \dots \cup D_m$ for $i \in I_j, j \in J$.
2. $\langle\langle R_m \rangle\rangle_\infty$ is the set of relations Q : There is a partition I_1, \dots, I_t of $[\text{ar}(Q)]$ and a function $\varphi : [t] \rightarrow [m]$ such that a tuple \mathbf{a} belongs to Q if and only if for each $j \in [t]$ and every $i, i' \in I_j$ the entries $\mathbf{a}[i], \mathbf{a}[i']$ are R_m -related and $\mathbf{a}[i] \in D_{\varphi(j)} \cup \dots \cup D_m$ for $i \in I_j, j \in J$.
3. $\langle R_m \rangle_{\max} = \langle R_m \rangle_{\max}^1$ is the set of relations Q : There is a partition I_1, \dots, I_t of $[\text{ar}(Q)]$ and $J \subseteq [t]$ such that a tuple \mathbf{a} belongs to Q if and only if for each $j \in [t]$ and every $i, i' \in I_j$ the entries $\mathbf{a}[i], \mathbf{a}[i']$ are R_m -related and $\mathbf{a}[i] \in D_m$ for $i \in I_j, j \in J$.

A set Γ such that $\langle \Gamma \rangle_k \neq \langle\langle \Gamma \rangle\rangle_k$ can be easily found among usual weak co-clones. For instance, for any weak co-clone Γ that is not a co-clone we have $\langle \Gamma \rangle_1 \neq \langle\langle \Gamma \rangle\rangle_1$. Such a weak co-clone can be found in, say, [22].

In the example given we have $\langle R_m \rangle_{\max}^1 = \langle R_m \rangle_m$. However, since $\langle R_{m-1} \rangle_m = \langle R_{m-1} \rangle$, we have $\langle R_{m-1} \rangle_{\max}^1 \neq \langle R_{m-1} \rangle_m$. For an example distinguishing between $\langle \Gamma \rangle_{\max}$ and $\langle \Gamma \rangle_{\max}^1$ see the Conclusion.

We give a sketchy proof of (1) here, the remaining results are similar. Let $Q(x_1, \dots, x_n)$ satisfies the conditions in (1) for a partition I_1, \dots, I_t of $[n]$ and $J \subseteq [t]$. Without loss of generality assume $J = [s]$, $s \leq t$. Choose variables $y_1, \dots, y_s \notin \{x_1, \dots, x_n\}$ and consider relation $S(x_1, \dots, x_n, y_1, \dots, y_s)$ given by: $\mathbf{a} \in S$ if and only if $(\mathbf{a}[i], \mathbf{a}[j]) \in R_m$ for any $i, j \in I_\ell$ for some $\ell \in [t]$ and $(\mathbf{a}[i], \mathbf{a}[n + \ell]) \in R_m$ for any $i \in I_\ell$ where $\ell \in J$. Clearly, $S \in \langle R_m \rangle = \langle\langle R_m \rangle\rangle$. Now, as it is easy to see,

$$Q(x_1, \dots, x_n) = \exists_k y_1 \dots \exists_k y_s S(x_1, \dots, x_n, y_1, \dots, y_s).$$

In order to show that every relation from $\langle\langle R_m \rangle\rangle_k$ satisfies these conditions, it suffices to prove that the set of relations Γ satisfying them is closed under manipulations with variables, conjunction, existential quantification, and k -existential quantification. The first three operations are easy, since Γ is a co-clone generated by R_m and unary relation $D' = D_k \cup \dots \cup D_m$. Let $Q(x_1, \dots, x_n) \in \Gamma$ and $S(x_1, \dots, x_{n-1}) = \exists_k x_n Q(x_1, \dots, x_n)$. Let also I_1, \dots, I_t and $J \subseteq [t]$ be the partition and a set from conditions (1). We may assume $n \in I_t$. Then if $t \in J$ then $S(x_1, \dots, x_{n-1}) = \exists x_n Q(x_1, \dots, x_n)$. Otherwise $\mathbf{a} \in S$ if and only if (a) for any $i, j \in I_\ell$, $\ell < t$, we have $(\mathbf{a}[i], \mathbf{a}[j]) \in R_m$, (b) for any $i, j \in I_t' = I_t - \{n\}$, we have $(\mathbf{a}[i], \mathbf{a}[j]) \in R_m$, and (c) $\mathbf{a}[i] \in D'$, whenever $i \in I_t' \cup \bigcup_{s \in J} I_s$. Therefore $S \in \langle\langle R_m \rangle\rangle_k$.

5 GALOIS CORRESPONDENCE

Let D be a finite set. A (partial) function $f: D^n \rightarrow D$ is said to be k -subset surjective if for any k -element subsets $A_1, \dots, A_n \subseteq D$ the image $f(A_1, \dots, A_n)$ has cardinality at least k . A (partial) function that is k -subset surjective for each k , $1 \leq k \leq |D|$ is said to be *subset surjective*. The set of all arity n k -subset surjective partial functions [arity n k -subset surjective functions, subset surjective functions] on D will be denoted by $P_D^{k,(n)}$ [resp., $F_D^{k,(n)}$, $F_D^{(n)}$]; furthermore, $P_D^k = \bigcup_{n \geq 0} P_D^{k,(n)}$, $F_D^k = \bigcup_{n \geq 0} F_D^{k,(n)}$, $F_D = \bigcup_{n \geq 0} F_D^{(n)}$. Any partial function is 1-subset surjective, while $|D|$ -subset surjective partial functions are exactly the surjective partial functions. Observe that this definition can be strengthened by allowing the sets A_i , $i \in [n]$, to have at least k elements.

Lemma 9 *If an n -ary function f is k -subset surjective, then for any subsets $A_1, \dots, A_n \subseteq D$ with $|A_i| \geq k$, $i \in [n]$, the image $f(A_1, \dots, A_n)$ has cardinality at least k .*

Proof: Choose any $B_i \subseteq A_i$, $i \in [n]$, and set $B = f(B_1, \dots, B_n)$. As f is k -subset surjective, $|B| \geq k$. Finally, $B \subseteq f(A_1, \dots, A_n)$, and the result follows. \square

The conditions of being k -subset surjective for different k are in general incomparable, as the following example shows.

Example 10 Let $D = \{0, \dots, k-1\}$ be a k -element set and $1 < m \leq k$. Then the following function f is not m -subset surjective, but is ℓ -subset surjective for any $\ell \in [k]$ except $\ell = m$. Function f is binary and given by its operation table:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 & m & \dots & k-1 \\ 1 & 1 & \dots & 1 & 2 & m & \dots & k-1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ m-3 & m-3 & \dots & m-3 & m-2 & m & \dots & k-1 \\ m-2 & m-2 & \dots & m-2 & 0 & m & \dots & k-1 \\ 0 & 1 & \dots & m-2 & 0 & m & \dots & k-1 \\ 0 & 1 & \dots & m-2 & m-1 & m & \dots & k-1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & \dots & m-2 & m-1 & m & \dots & k-1 \end{pmatrix}.$$

Clearly, f is not m -subset surjective, because $f(B, B) = \{0, \dots, m-2\}$ for $B = \{0, \dots, m-1\}$. Also, as it is a total function, f is 1-subset surjective. Take $\ell \in [k]$, $\ell > 1$, and $B_1, B_2 \subseteq \{0, \dots, k-1\}$ with $|B_1| = |B_2| = \ell$. If there is $a \in B_1$ with $i \geq m$ then $f(a, b_1) \neq f(a, b_2)$ whenever $b_1 \neq b_2$. This means that $|f(B_1, B_2)| \geq \ell$ in this case, and, in particular, f is ℓ -subset surjective for any $\ell > m$. So, suppose $\ell < m$ and $B_1 \subseteq \{0, \dots, m-1\}$. If $B_1 \subseteq \{0, \dots, m-2\}$ then take $b \in B_2 \cap \{0, \dots, m-2\}$ and observe that $f(a_1, b) \neq f(a_2, b)$ for any $a_1, a_2 \in \{0, \dots, m-2\}$, $a_1 \neq a_2$. Thus, $|f(B_1, \{b\})| = \ell$. Suppose $m-1 \in B_1$. If $B_2 \subseteq \{0, \dots, m-2\}$, then $|f(m-1, B_2)| = \ell$; assume $m-1 \in B_2$. As is easily seen, $B_1 \cap \{0, \dots, m-2\} \subseteq f(B_1, B_2)$. There is $a \in \{0, \dots, m-2\}$ such that $a \notin B_1$ but $a-1 \pmod{m-1} \in B_1$. Then $a \in f(B_1, B_2)$, since $a = f(a-1, m-1)$. Thus, $|f(B_1, B_2)| \geq \ell$.

The notion of invariance for k -subset surjective functions is the standard one for partial functions and relations. As usual, if C is a set of (k -) subset surjective (partial) functions, $\text{Inv}(C)$ denotes the set of relations invariant with respect to every function from C . For a set Γ of relations, $\text{m}(k)\text{-Pol}(\Gamma)$ and $\text{m}(k)\text{-pPol}(\Gamma)$ denote the set of all k -subset surjective functions and partial functions, respectively, preserving every relation from Γ . For a set $K \subseteq \mathbb{N}$ by $\text{m}(K)\text{-Pol}(\Gamma)$ and $\text{m}(K)\text{-pPol}(\Gamma)$ we denote the set of all functions and, respectively, partial functions preserving every relation from Γ that are k -subset surjective for each $k \in K$. Thus, in particular,

$$\text{m}(K)\text{-Pol}(\Gamma) = \bigcap_{k \in K} \text{m}(k)\text{-Pol}(\Gamma), \quad \text{and} \quad \text{m}(K)\text{-pPol}(\Gamma) = \bigcap_{k \in K} \text{m}(k)\text{-pPol}(\Gamma).$$

By $m\text{-Pol}(\Gamma)$ we denote the analogous set of subset surjective functions.

The operator Inv on one side and the operators $m(k)\text{-pPol}(\Gamma)$, $m(k)\text{-Pol}(\Gamma)$, $m(K)\text{-Pol}(\Gamma)$, $m\text{-pPol}(\Gamma)$, $m\text{-Pol}(\Gamma)$ on the other side form Galois correspondences in the standard fashion. We characterize closed sets of relations that give rise from this correspondence.

Lemma 11 *Let $R(x_1, \dots, x_\ell, y)$ be a relation on D , and let $Q(x_1, \dots, x_\ell) = \exists_k y R(x_1, \dots, x_\ell, y)$. Then if a k -subset surjective (partial) function f preserves R , it also preserves Q .*

Proof: Suppose f is n -ary. Take $\mathbf{a}_1, \dots, \mathbf{a}_n \in Q$. Since each of them is put into Q by k -existential quantification, it has at least k extensions to a tuple from R . Let $B_1, \dots, B_n \subseteq D$ be such that $|B_i| \geq k$ and $(\mathbf{a}_i, b) \in R$ for $b \in B_i$ and $i \in [n]$. Let also $\mathbf{b} = f(\mathbf{a}_1, \dots, \mathbf{a}_n)$. For any $b \in B = f(B_1, \dots, B_n)$ the tuple (\mathbf{b}, b) belongs to R . As f is k -subset surjective, $|B| \geq k$, hence, $\mathbf{b} \in Q$. \square

Theorem 12 *Let Γ be a set of relations on a set D and $K \subseteq \mathbb{N}$. Then $\text{Inv}(m(K)\text{-pPol}(\Gamma)) = \langle \Gamma \rangle_K$.*

Proof: We will assume that $K = \{k_1, \dots, k_s\} \subseteq \{1, \dots, |D|\}$. Indeed, if $k \geq |D|$ then $\exists_k x R$ is empty for any relation on D . The equality relation, $=_D$, is invariant with respect to any partial function on D . Let f be a k -subset surjective functions. It is straightforward to verify that manipulations of variables of a predicate invariant under f and the conjunction of any two predicates invariant under f result in predicates invariant under f , again, since it is true for any partial function. By Lemma 11 applying k -quantification to a predicate invariant under f gives a predicate invariant under f , again because it is true for any partial function. Hence, $\langle \Gamma \rangle_K \subseteq \text{Inv}(m(K)\text{-pPol}(\Gamma))$. Moreover, it follows that $\text{Inv}(m(K)\text{-pPol}(\Gamma)) = \text{Inv}(m(K)\text{-pPol}(\langle \Gamma \rangle_K))$.

To establish the reverse inclusion, take an ℓ -ary relation $R \in \text{Inv}(m(K)\text{-pPol}(\Gamma))$. We need to show that $R \in \langle \Gamma \rangle_K$. Define a relation Q as follows. Let $R = \{\mathbf{a}_1, \dots, \mathbf{a}_t\}$. For each $k \in K$ we consider sequences (B_1, \dots, B_t) of k -element subsets of D . Let also $(B_1^{k_1}, \dots, B_t^{k_1}), \dots, (B_1^{k_{r_k}}, \dots, B_t^{k_{r_k}})$ be a list of all such sequences. Let S_k^j be the relation

$$\underbrace{B_j^{k_1} \times \dots \times B_j^{k_1}}_{k \text{ times}} \times \dots \times \underbrace{B_j^{k_{r_k}} \times \dots \times B_j^{k_{r_k}}}_{k \text{ times}},$$

and $S^j = S_{k_1}^j \times \dots \times S_{k_s}^j$. Then Q is the union of relations given by $\mathbf{a}_j \times S^j$, for all $j \in [t]$. We show that there is $S \in \langle \Gamma \rangle_K$ such that $Q \subseteq S$ and $\text{pr}_{[\ell]} S = R$. Then applying k -quantifications, $k \in K$, to all coordinates of S except for the first ℓ we infer that $R \in \langle \Gamma \rangle_K$.

Set $M = \sum_{k \in K} k r_k$ and $M_j = \sum_{i=1}^j k_i r_{k_i}$; by N_K , $k \in K$, we denote the set $\{M_j + 1, \dots, M_{j+1}\}$. Let us consider the relation $S = \bigcap \{Q' \in \langle \Gamma \rangle_K \mid Q \subseteq Q'\}$. Since $\langle \Gamma \rangle_K$ is closed under conjunctions and contains the total relation $D^{\ell+M}$, we have $S \in \langle \Gamma \rangle_K$ and $Q \subseteq S$.

Now choose any tuple $\mathbf{b} = (b_1, \dots, b_\ell, d_1, \dots, d_M) \in S$. There are sets C_1, \dots, C_M such that $|C_i| = k_j$, $i \in [M]$, whenever $i \in N_j$, for any $t \in [r_j]$, $C_{M_{j-1}+k_j(t-1)+1} = \dots = C_{M_{j-1}+k_j t}$, $d_i \in C_i$, and for any $d'_i \in C_i$, $i \in [M]$, the tuple $(b_1, \dots, b_\ell, d'_1, \dots, d'_M) \in S$. Indeed, otherwise we can applying a sequence of k -quantifications for $k \in K$ to obtain an ℓ -ary relation S' containing R , but not (b_1, \dots, b_ℓ) . Then $(S' \times D^{\ell+M}) \cap Q$ belongs to $\langle \Gamma \rangle_K$, but is smaller than Q . Therefore we can choose \mathbf{b} such that for any $j \in [s]$ and any $t \in [r_j]$ all the values $d_{M_{j-1}+k_j(t-1)+1}, \dots, d_{M_{j-1}+k_j t}$ are distinct, and $\{d_{M_{j-1}+k_j(t-1)+1}, \dots, d_{M_{j-1}+k_j t}\} = C_{M_{j-1}+k_j t}$.

Since $\langle \Gamma \rangle_K$ is closed under conjunctions, by the Fleischer and Rosenberg result [20] it satisfies $\langle \Gamma \rangle_K = \text{Inv}(\text{pPol}(\langle \Gamma \rangle_K))$. Moreover, by the proof of Theorem 2 of [20] S is the set of all tuples of the form $f(\mathbf{c}_1, \dots, \mathbf{c}_n)$ for $n \geq 1$, $\mathbf{c}_1, \dots, \mathbf{c}_n \in Q$, and $f \in \text{pPol}(\langle \Gamma \rangle_K)$. Therefore there exist $n \geq 1$, $\mathbf{c}_1, \dots, \mathbf{c}_n \in Q$ and $f \in \text{pPol}(\langle \Gamma \rangle_K)$ such that $\mathbf{b} = f(\mathbf{c}_1, \dots, \mathbf{c}_n)$. Let $\text{pr}_{[\ell]} \mathbf{c}_q = \mathbf{a}_{i_q}$. For any selection E_1, \dots, E_n of k_j -element subsets of D , $j \in [s]$, there is $t \in [r_{k_j}]$ such that $E_q = B_{i_q}^{k_j t}$ for $q \in [n]$. By the choice of \mathbf{b} the range of f on $E_1 \times \dots \times E_n = B_{i_1}^{k_{j_1} t} \times \dots \times B_{i_n}^{k_{j_n} t}$ contains $C_{M_{j-1}+k_j t}$. Hence f is k_j -subset surjective for any $k_j \in K$, and so $f \in m(K)\text{-pPol}(\Gamma)$, as it is equal

to $m(K)\text{-pPol}(\langle\Gamma\rangle_k)$. Therefore R is invariant under f , and so $(b_1, \dots, b_\ell) \in R$. Relation S satisfies the required conditions, which completes the proof. \square

Corollary 13 *There is a Galois correspondence between K -existential partial co-clones on one side and partial clones generated by K -surjective partial functions on the other side.*

More precisely, for any set Γ of relations on D , any $K \subseteq \{1, \dots, |D|\}$, and any set C of K -surjective partial functions on D ,

- $\text{Inv}(C)$ is a K -existential partial co-clone;
- $\text{pPol}(\Gamma)$ is a partial co-clone generated by the set $m(K)\text{-pPol}(\langle\Gamma\rangle_K)$ of K -surjective partial functions;
- $\text{Inv}(m(K)\text{-pPol}(\Gamma)) = \langle\Gamma\rangle_K$;
- $m(K)\text{-pPol}(\text{Inv}(C))$ is the set of K -surjective functions from the partial clone generated by C .

Corollary 14 *Let Γ be a set of relations on a set D .*

- (a) $\text{Inv}(m(k)\text{-pPol}(\Gamma)) = \langle\Gamma\rangle_k$;
- (b) $\text{Inv}(m(k)\text{-Pol}(\Gamma)) = \langle\langle\Gamma\rangle\rangle_k$;
- (c) $\text{Inv}(m\text{-Pol}(\Gamma)) = \langle\langle\Gamma\rangle\rangle_\infty$;

6 THE LATTICE OF BOOLEAN MAX-CO-CLONES

In this section we give a description of all max-co-clones on $\{0, 1\}$. We will use the description of usual Boolean co-clones from [28] and *plain bases* of Boolean co-clones found in [14]. Recall that plain basis of a co-clone C is a set Γ of relations such that the closure of Γ with respect to manipulation of variables and conjunction is C .

To state the results of [14] and then to proceed with the proof, we need some definitions and notation. A relation $R(x_1, \dots, x_n)$ is said to be *trivial* if it can be specified by giving a set of variables that are equal to 0 (to 1) in every tuple from R , and a collection of conditions of the form $x_i = x_j$. More formally, there are sets $Z, W \subseteq [n]$ and an equivalence relation \sim on $[n] - (Z \cup W)$ such that $\mathbf{a} \in R$ if and only if $\mathbf{a}[i] = 0$ whenever $i \in Z$, $\mathbf{a}[i] = 1$ whenever $i \in W$, and $\mathbf{a}[i] = \mathbf{a}[j]$ whenever $i \sim j$. A relation is called *monotone* if it is invariant with respect to \vee , the Boolean disjunction operation, or \wedge , the Boolean conjunction operation. Relation R is called *self-complement* if along with any tuple $\mathbf{a} \in R$ it also contains its *complement*, the tuple $\neg\mathbf{a}$ such that $\neg\mathbf{a}[i] = 1$ if and only if $\mathbf{a}[i] = 0$. Finally, relation R is called *affine* if it is the set of solutions to a system of linear equations over $GF(2)$. Addition in $GF(2)$ we denote by \oplus .

For $I \subseteq [n]$ we denote by \mathbf{a}_I the assignment to x_1, \dots, x_n in which $\mathbf{a}[i] = 1$ if $i \in I$ and $\mathbf{a}[i] = 0$ otherwise. We will use the following notation: δ_0, δ_1 denote the unary *constant* relations $\{(0)\}, \{(1)\}$, respectively. EQ is the binary *equality* relation $\{(0, 0), (1, 1)\}$; while NEQ is the binary *disequality* relation $\{(0, 1), (1, 0)\}$. $\text{IMP}^k(x_1, \dots, x_k, y)$ is the Horn $(k+1)$ -ary relation given by the formula $\neg x_1 \vee \dots \vee \neg x_k \vee y$, that is, $\mathbf{a} \in R$ if and only if $(\mathbf{a}[1], \dots, \mathbf{a}[k], \mathbf{a}[k+1])$ satisfies the formula. By NIMP^k we denote the anti-Horn relation given by the formula $x_1 \vee \dots \vee x_k \vee \neg y$. OR^k denotes the relation $\{0, 1\}^k - \{(0, \dots, 0)\}$, and NAND^k denotes the relation $\{0, 1\}^k - \{(1, \dots, 1)\}$. Finally, $\text{Compl}_{k,\ell}$ is the $(k+\ell)$ -ary relation $\{0, 1\}^{k+\ell} - \{(0, \dots, 0, 1, \dots, 1), (1, \dots, 1, 0, \dots, 0)\}$, where the first of the two excluded tuples contains k zeros and ℓ ones, while the second contains k ones and ℓ zeros.

Fig. 1 shows the lattice of Boolean co-clones (borrowed from [14]), and Table 1 lists plain bases of Boolean co-clones. Table 1 is also taken from [14] only with notation changed to match the one used here.

The next theorem states the main result of this section.

Co-clone	Plain basis
IBF	$\{EQ\}$
IR_0	$\{EQ, \delta_0\}$
IR_1	$\{EQ, \delta_1\}$
IR_2	$\{EQ, \delta_0, \delta_1\}$
IM	$\{IMP\}$
IM_0	$\{IMP, \delta_0\}$
IM_1	$\{IMP, \delta_1\}$
IM_2	$\{IMP, \delta_0, \delta_1\}$
IS_0^k	$\{EQ\} \cup \{OR^\ell \mid \ell \leq k\}$
IS_0	$\{EQ\} \cup \{OR^\ell \mid \ell \in \mathbb{N}\}$
IS_1^k	$\{EQ\} \cup \{NAND^\ell \mid \ell \leq k\}$
IS_1	$\{EQ\} \cup \{NAND^\ell \mid \ell \in \mathbb{N}\}$
IS_{02}^k	$\{EQ, \delta_0\} \cup \{OR^\ell \mid \ell \leq k\}$
IS_{02}	$\{EQ, \delta_0\} \cup \{OR^\ell \mid \ell \in \mathbb{N}\}$
IS_{12}^k	$\{EQ, \delta_1\} \cup \{NAND^\ell \mid \ell \leq k\}$
IS_{12}	$\{EQ, \delta_1\} \cup \{NAND^\ell \mid \ell \in \mathbb{N}\}$
IS_{01}^k	$\{IMP\} \cup \{OR^\ell \mid \ell \leq k\}$
IS_{01}	$\{IMP\} \cup \{OR^\ell \mid \ell \in \mathbb{N}\}$
IS_{11}^k	$\{IMP\} \cup \{NAND^\ell \mid \ell \leq k\}$
IS_{11}	$\{IMP\} \cup \{NAND^\ell \mid \ell \in \mathbb{N}\}$
IS_{00}^k	$\{IMP, \delta_0\} \cup \{OR^\ell \mid \ell \leq k\}$
IS_{00}	$\{IMP, \delta_0\} \cup \{OR^\ell \mid \ell \in \mathbb{N}\}$
IS_{10}^k	$\{IMP, \delta_1\} \cup \{NAND^\ell \mid \ell \leq k\}$
IS_{10}	$\{IMP, \delta_1\} \cup \{NAND^\ell \mid \ell \in \mathbb{N}\}$
ID	$\{EQ, NEQ\}$
ID_1	$\{EQ, NEQ, \delta_0, \delta_1\}$
ID_2	$\{\delta_0, \delta_1, OR, IMP, NAND\}$
IL	$\{x_1 \oplus \dots \oplus x_k = 0 \mid k \text{ even}\}$
IL_0	$\{x_1 \oplus \dots \oplus x_k = 0 \mid k \in \mathbb{N}\}$
IL_1	$\{x_1 \oplus \dots \oplus x_k = c \mid k \in \mathbb{N}, k \equiv c \pmod{2}, c \in \{0, 1\}\}$
IL_2	$\{x_1 \oplus \dots \oplus x_k = c \mid k \in \mathbb{N}, c \in \{0, 1\}\}$
IL_3	$\{x_1 \oplus \dots \oplus x_k = c \mid k \text{ even}, c \in \{0, 1\}\}$
IV	$\{IMP^k \mid k \geq 1\}$
IV_0	$\{IMP^k \mid k \geq 1\} \cup \{\delta_0\}$
IV_1	$\{OR^k \mid k \in \mathbb{N}\} \cup \{IMP^k \mid k \geq 1\}$
IV_2	$\{OR^k \mid k \in \mathbb{N}\} \cup \{IMP^k \mid k \geq 1\} \cup \{\delta_0\}$
IE	$\{NIMP^k \mid k \geq 1\}$
IE_0	$\{NAND^k \mid k \in \mathbb{N}\} \cup \{NIMP^k \mid k \geq 1\}$
IE_1	$\{NIMP^k \mid k \geq 1\} \cup \{\delta_1\}$
IE_2	$\{NAND^k \mid k \in \mathbb{N}\} \cup \{NIMP^k \mid k \geq 1\} \cup \{\delta_1\}$
IN	$\{Compl_{k,\ell} \mid k, \ell \geq 1\}$
IN_2	$\{Compl_{k,\ell} \mid k, \ell \in \mathbb{N}\}$
II	$\{x_1 \vee \dots \vee x_k \vee \neg y_1 \vee \dots \vee \neg x_\ell \mid k, \ell \geq 1\}$
II_0	$\{x_1 \vee \dots \vee x_k \vee \neg y_1 \vee \dots \vee \neg x_\ell \mid k, \ell \geq 1\} \cup \{\delta_0\}$
II_1	$\{x_1 \vee \dots \vee x_k \vee \neg y_1 \vee \dots \vee \neg x_\ell \mid k, \ell \geq 1\} \cup \{\delta_1\}$
II_2	$\{x_1 \vee \dots \vee x_k \vee \neg y_1 \vee \dots \vee \neg x_\ell \mid k, \ell \geq 1\} \cup \{\delta_0, \delta_1\}$

TABLE 1
Plain bases of Boolean co-clones

Max-co-clone	Max-basis
IBF	$\{EQ\}$
IR_0	$\{EQ, \delta_0\}$
IR_1	$\{EQ, \delta_1\}$
IR_2	$\{EQ, \delta_0, \delta_1\}$
IM_2	$\{IMP\}$
IS_0^k	$\{EQ\} \cup \{OR^k\}$
IS_0	$\{EQ\} \cup \{OR^\ell \mid \ell \in \mathbb{N}\}$
IS_1^k	$\{EQ\} \cup \{NAND^k\}$
IS_1	$\{EQ\} \cup \{NAND^\ell \mid \ell \in \mathbb{N}\}$
IS_{02}^k	$\{EQ, \delta_0, OR^k\}$
IS_{02}	$\{EQ, \delta_0\} \cup \{OR^\ell \mid \ell \in \mathbb{N}\}$
IS_{12}^k	$\{EQ, \delta_1\} \cup \{NAND^\ell \mid \ell \leq k\}$
IS_{12}	$\{EQ, \delta_1\} \cup \{NAND^\ell \mid \ell \in \mathbb{N}\}$
ID	$\{EQ, NEQ\}$
ID_1	$\{EQ, NEQ, \delta_0, \delta_1\}$
IL	$\{x_1 \oplus \dots \oplus x_k = 0 \mid k \text{ even}\}$
IL_0	$\{x_1 \oplus \dots \oplus x_k = 0 \mid k \in \mathbb{N}\}$
IL_1	$\{x_1 \oplus \dots \oplus x_k = c \mid k \in \mathbb{N}, k \equiv c \pmod{2}, c \in \{0, 1\}\}$
IL_2	$\{x_1 \oplus \dots \oplus x_k = c \mid k \in \mathbb{N}, c \in \{0, 1\}\}$
IL_3	$\{x_1 \oplus \dots \oplus x_k = c \mid k \text{ even}, c \in \{0, 1\}\}$
IN_2	$\{Compl_{3,0}\}$
II_2	$\{IMP, OR\}$

TABLE 2
Max-bases of Boolean max-co-clones

Theorem 15 *The lattice of Boolean max-co-clones is shown in Fig 2. Some generating sets of these max-co-clones are given in Table 2.*

The theorem will follow from a sequence of auxiliary statements. In Section 6.1 we show that using the \exists_{\max} quantifier we can define various relations, and that any relation can be defined by any two nontrivial binary relations. Then we show, Lemma 19, that any proper max-co-clone must contain only monotone, or only self-complement, or only affine relations. We consider these three cases. In the case of affine relations we show that the max-co-clones of such relations are exactly regular co-clones, Lemma 21. Then we show, Proposition 30, that there is only one max-co-clone of self-complement relations, which contains a non-affine relation, IN_2 . Then we show, Lemmas 23,24, that there is only one proper, that is, not II_2 , the set of all relations, max-co-clone containing IMP, and this max-co-clone is IM_2 . Finally, we consider the four remaining infinite chains of co-clones. In Lemma 25 we introduce a property that defines them. Then we show, Lemma 26, and 28, that there are no other max-co-clones containing OR (for NAND a dual result holds). Finally, we show that each of these co-clones is a max-co-clone.

6.1 Some implementations

We start with several useful observations.

- Lemma 16** (1) $\delta_0, \delta_1 \in \langle \text{IMP} \rangle_{\max}$;
(2) $\delta_0 \in \langle \text{NEQ}, \delta_1 \rangle_{\max}$, $\delta_1 \in \langle \text{NEQ}, \delta_0 \rangle_{\max}$;
(3) $\text{NAND}^k \in \langle \text{NAND}^m \rangle_{\max}$ for any $k \leq m$;
(4) $\text{OR}^k \in \langle \text{OR}^m \rangle_{\max}$ for any $k \leq m$.

- Proof:** (1) As is easily seen, $\delta_0(x) = \exists_{\max} y \text{IMP}(x, y)$, and $\delta_1(x) = \exists_{\max} y \text{IMP}(y, x)$.
(2) The first inclusion follows from $\delta_0(x) = \exists_{\max} y (\text{NEQ}(x, y) \wedge \delta_1(y))$; the second one is similar.
(3) This claim follows from $\text{NAND}^{m-1}(x_1, \dots, x_{m-1}) = \exists_{\max} x_m \text{NAND}^m(x_1, \dots, x_m)$.
(4) is similar to (3). □

Lemma 17 *For any two different relations $R, R' \in \{\text{NEQ}, \text{IMP}, \text{OR}, \text{NAND}\}$, $\langle R, R' \rangle_{\max} = II_2$, the set of all relations on $\{0, 1\}$.*

Proof: Observe first that

$$\begin{aligned} \text{OR} \cap \text{NAND} &= \text{NEQ}, \\ \text{IMP}(x, y) &= \exists_{\max} z (\text{OR}(z, y) \wedge \text{NEQ}(z, x)) \\ &= \exists_{\max} z (\text{NAND}(x, z) \wedge \text{NEQ}(z, y)) \\ \text{OR}(x, y) &= \exists_{\max} z (\text{IMP}(z, y) \wedge \text{NEQ}(z, x)) \\ &= \exists_{\max} z, t (\text{NAND}(z, t) \wedge \text{NEQ}(z, x) \wedge \text{NEQ}(t, y)) \\ \text{NAND}(x, y) &= \exists_{\max} z (\text{IMP}(x, z) \wedge \text{NEQ}(z, x)) \\ &= \exists_{\max} z, t (\text{OR}(z, t) \wedge \text{NEQ}(z, x) \wedge \text{NEQ}(t, y)). \end{aligned}$$

Also in the relation $Q(x, y, z, t) = \text{OR}(x, y) \wedge \text{IMP}(x, z) \wedge \text{IMP}(y, t)$ assignments $(0, 1)$ and $(1, 0)$ to x, y are extendible in two ways, while $(1, 1)$ is extendible in only one way. Therefore

$$\begin{aligned} \text{NEQ}(x, y) &= \exists_{\max} (z, t) (\text{OR}(x, y) \wedge \text{IMP}(x, z) \wedge \text{IMP}(y, t)), \text{ and, similarly,} \\ \text{NEQ}(x, y) &= \exists_{\max} (z, t) (\text{NAND}(x, y) \wedge \text{IMP}(z, x) \wedge \text{IMP}(t, y)). \end{aligned}$$

Thus $\{\text{NEQ}, \text{IMP}, \text{OR}, \text{NAND}\} \subseteq \langle R, R' \rangle_{\max}$, and it suffices to show that $\langle \text{NEQ}, \text{IMP}, \text{OR}, \text{NAND} \rangle_{\max} = II_2$.

The rest of the proof is derived from that of Lemma 15 [11], only it does not have to deal with weights.

Let $R(x_1, \dots, x_n)$ be any relation. For each $I \subseteq [n]$ with $\mathbf{a}_I \in R$ introduce a new variable z_I . Consider the relation given by

$$Q = \bigwedge_{I \subseteq [n], \mathbf{a}_I \in R} \left(\bigwedge_{i \in I} \text{IMP}(z_I, x_i) \wedge \bigwedge_{i \notin I} \text{NAND}(z_I, x_i) \right).$$

Every assignment $\mathbf{a}_I \in R$ can be extended to the variables z_I in two ways: with $z_I = 0$ and $z_I = 1$. Any other assignment can be extended in only one way. Therefore

$$R(x_1, \dots, x_n) = \exists_{\max}(z_I)_{I \subseteq [n], \mathbf{a}_I \in R} Q,$$

which completes the proof. \square

Lemma 18 *Let R be a non-affine relation and $a \in \{0, 1\}$. Then $\langle R, \text{NEQ}, \delta_a \rangle_{\max} = II_2$.*

Proof: By Lemma 17 it suffices to prove that one of IMP, OR, or NAND belongs to $\langle f, \text{NEQ}, \delta_a \rangle_{\max}$. Observe first that we can always assume that the all-zero tuple $\mathbf{a}_\emptyset \in R$. Indeed, if for some $I \subseteq [n]$ we have $\mathbf{a}_I \in R$ then the relation

$$R'(x_1, \dots, x_n) = \exists_{\max}(z_i)_{i \in I} \left(R(x_1, \dots, x_n) \wedge \bigwedge_{i \in I} \text{NEQ}(z_i, x_i) \right)$$

contains \mathbf{a}_\emptyset . As $R \notin IL_2$, by Lemma 4.10 of [15], there are tuples $\mathbf{a}, \mathbf{b}, \mathbf{c} \in R$ such that $\mathbf{d} = \mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \notin R$. Observing that $\mathbf{e} \in R$ if and only if $\mathbf{e} \oplus \mathbf{a}_I \in R'$, we have that $\mathbf{a} \oplus \mathbf{a}_I, \mathbf{b} \oplus \mathbf{a}_I, \mathbf{c} \oplus \mathbf{a}_I \in R'$, but $\mathbf{d} \oplus \mathbf{a}_I = (\mathbf{a} \oplus \mathbf{a}_I) \oplus (\mathbf{b} \oplus \mathbf{a}_I) \oplus (\mathbf{c} \oplus \mathbf{a}_I) \notin R$. Hence R' is not affine as well. Also, if $b \in \{0, 1\}$ is such that $\{0, 1\} = \{a, b\}$ then by Lemma 16(2) $\delta_0, \delta_1 \in \langle R, \text{NEQ}, \delta_a \rangle_{\max}$.

Again we use Lemma 4.10 of [15] to find to find tuples $\mathbf{a}, \mathbf{b}, \mathbf{c} \in R$ such that $\mathbf{d} = \mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \notin R$. Note that \mathbf{a} can be chosen to be the all-zero tuple \mathbf{a}_\emptyset . After rearranging variables these tuples can be represented as follows

a	0...0	0...0	0...0	0...0	$\in R$
b	0...0	0...0	1...1	1...1	$\in R$
c	0...0	1...1	0...0	1...1	$\in R$
d	0...0	1...1	1...1	0...0	$\notin R$
	$x \dots x$	$y \dots y$	$z \dots z$	$t \dots t$	

Denote by R' the relation obtained from R by identifying variables as shown in the last row of the table. Relation R' contains tuples $(0, 0, 0, 0), (0, 0, 1, 1), (0, 1, 0, 1)$ but does not contain $(0, 1, 1, 0)$, and so does not belong to IL_2 . Replacing R' with

$$R''(x, y, z) = \exists_{\max} t (R(t, x, y, z) \wedge \delta_0(t)),$$

we obtain a relation R'' such that $(0, 0, 0), (0, 1, 1), (1, 0, 1) \in R''$ but $(1, 1, 0) \notin R''$.

We now proceed depending on which of the 4 remaining tuples (a) $(1, 0, 0)$, (b) $(0, 1, 0)$, (c) $(0, 0, 1)$, and (d) $(1, 1, 1)$ relation R'' contains. If it contains none of (a)–(d) then $\text{NAND}(x, y) = \exists_{\max} z R''(x, y, z)$. If it contains (a) or (b) but not (d) then NAND is obtained by identifying y and z , or x and z , respectively. If R'' contains (c) but not (d) then $\text{NAND}(x, y) = \exists_{\max} z (R''(x, y, z) \wedge \delta_1(z))$. If it contains (d) but not (a) then $\text{IMP}(x, y) = R''(x, y, y)$. In the case R'' contains (a), (d), but does not contain (b) IMP is obtained by identifying x and z . If R'' contains (a), (d), and (b) $\text{OR}(x, y) = \exists_{\max} z (R''(x, y, z) \wedge \delta_1(z))$. Finally, if the relation contains all of (a)–(d) $\text{IMP}(y, x) = R''(y, y, x)$. \square

Next we show that every max-co-clone is a subset of IL_2, IN_2, IV_2 , or IE_2 .

Lemma 19 *Let Γ be a set of relations, which is not affine, monotone, or self-complement. Then $\langle \Gamma \rangle_{\max} = II_2$.*

Proof: Let $R(x_1, \dots, x_n) \in \Gamma$ be a non-self-complement relation. Then after suitable rearrangement of variables there is $i \in \{0, \dots, n\}$ such that $\mathbf{a}_{[i]} \in R$, while $\mathbf{a}_{[n]-[i]} \notin R$. If $0 < i < n$ then identifying variables x_1, \dots, x_i and x_{i+1}, \dots, x_n we obtain a binary relation R' that contains $(1, 0)$ but does not contain $(0, 1)$. As is easily seen either $\exists_{\max} x R'$ or $\exists_{\max} y R'$ is a constant relation. In the case $i = 0$ or $i = n$, identifying all variables of R we obtain a constant relation. Thus either $\delta_0 \in \langle \Gamma \rangle_{\max}$ or $\delta_1 \in \langle \Gamma \rangle_{\max}$.

Suppose $\delta_1 \in \langle \Gamma \rangle_{\max}$. The case $\delta_0 \in \langle \Gamma \rangle_{\max}$ is similar. By Lemma 5.30 of [15] for any non-affine relation $R \in \Gamma$, the set $\langle R, \delta_1 \rangle \subseteq \langle R, \delta_1 \rangle_{\max}$ contains one of the following relations: OR, IMP, NAND. If $\text{NAND} \in \langle R, \delta_1 \rangle_{\max}$ then $\delta_0(x) = \text{NAND}(x, x)$, and we can make all the arguments below for δ_0 and NAND. Therefore we have two cases to consider. Suppose first that $\text{OR} \in \langle R, \delta_1 \rangle_{\max}$. There is a relation $Q \in \Gamma$ that is not invariant under the \vee operation. Therefore for some tuple $\mathbf{a}, \mathbf{b} \in Q$ the tuple $\mathbf{a} \vee \mathbf{b}$ does not belong to Q . After an appropriate rearrangement of variables these tuples can be represented as follows

a	0...0	0...0	1...1	1...1	$\in Q$
b	0...0	1...1	0...0	1...1	$\in Q$
d	0...0	1...1	1...1	1...1	$\notin Q$
	$x \dots x$	$y \dots y$	$z \dots z$	$t \dots t$	

Denote by Q' the relation obtained from Q by identifying variables as shown in the last row of the table. Relation Q' contains tuples $(0, 0, 1, 1)$, $(0, 1, 0, 1)$ but does not contain $(0, 1, 1, 1)$. Then, relation $Q''(x, y, z) = \exists_{\max} t (Q'(x, y, z, t) \wedge \delta_1(t) \wedge \text{OR}(y, z))$ contains tuples $(0, 0, 1)$, $(0, 1, 0)$ but does not contain $(0, 1, 1)$, $(0, 0, 0)$, $(1, 0, 0)$. We have several cases depending on the 3 remaining tuples (a) $(1, 1, 0)$, (b) $(1, 0, 1)$, (c) $(1, 1, 1)$. If none of (a)–(c) is in Q'' then $\text{NEQ}(x, y) = \exists_{\max} z Q''(z, x, y)$. If Q'' contains (a) but not (c) (or (b) but not (c)), then $\text{NEQ}(x, y) = Q''(x, x, y)$ (respectively, $\text{NEQ}(x, y) = Q''(x, y, x)$). If it contains (c) but does not contain (a) and (b) then $\text{IMP}(x, y) = \exists_{\max} z Q''(x, y, z)$. If Q'' contains both (b) and (c) then $\text{IMP}(x, y) = \exists_{\max} z (Q''(x, y, z) \wedge \delta_1(z))$. Finally if Q'' contains (a), (c), but not (b), then $\text{IMP}(x, y) = \exists_{\max} z (Q''(y, z, x) \wedge \delta_1(z))$.

In either case $\langle \Gamma \rangle_{\max}$ contains a constant relation, either NEQ or IMP, and contains one of OR, IMP, NAND. If it contains NEQ, we are done by Lemma 17. So suppose $\text{IMP} \in \langle \Gamma \rangle_{\max}$. Then we also have $\delta_0, \delta_1 \in \langle \Gamma \rangle_{\max}$. Since Γ is not monotone, as before we can derive relations $S_1, S_2 \in \langle \Gamma \rangle_{\max}$ such that $(0, 0, 1, 1), (0, 1, 0, 1) \in S_1, S_2$, but $(0, 1, 1, 1) \notin S_1, (0, 0, 0, 1) \notin S_2$. Now it is easy to see that $\text{NEQ} = S'_1 \wedge S'_2$, where $S'_i(x, y) = \exists_{\max} z \exists_{\max} t (S_i(z, x, y, t) \wedge \delta_0(z) \wedge \delta_1(t))$. \square

6.2 Affine relations

Recall that the set of affine relations, that is, $(n\text{-ary})$ relations that can be represented as the set of solutions to a system of linear equations over $\text{GF}(2)$ is denoted by II_2 . The next lemma follows from basic linear algebra, as sets of extensions of tuples are cosets of the same vector subspace. For the sake of completeness we give a proof of this lemma.

Lemma 20 *Let R be an $(n\text{-ary})$ affine relation. Then for any $I \subseteq [n]$ any two tuples $\mathbf{a}, \mathbf{b} \in \text{pr}_I R$ have the same number of extensions to tuples from R .*

Proof: Let R be the set of solutions of a system of linear equations $A \cdot \mathbf{x} = \mathbf{c}$, where A is a $\ell \times n$ -matrix over $\text{GF}(2)$, $\mathbf{x} = (x_1, \dots, x_n)^\top$, and $\mathbf{c} \in \{0, 1\}^\ell$. Without loss of generality $I = [k]$. Then A can be represented as $A = [A_1 \mid A_2]$, where A_1 is a $\ell \times k$ -matrix and A_2 is a $\ell \times (n - k)$ -matrix; \mathbf{x} can be represented as $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2)^\top$, where $\mathbf{x}^1 = (x_1, \dots, x_k)$, $\mathbf{x}^2 = (x_{k+1}, \dots, x_n)$. Fix $\mathbf{a} \in \text{pr}_{[k]} R$ and set $\mathbf{c}_\mathbf{a} = \mathbf{c} \oplus (A_1 \cdot \mathbf{a})$. The set of extensions of \mathbf{a} is the set of solutions of the system $A_2 \cdot \mathbf{x}^2 = \mathbf{c}_\mathbf{a}$. Clearly, the number of solutions this system does not depend on \mathbf{a} , provided the system is consistent. \square

Lemma 21 Let $\Gamma \subseteq IL_2$. Then Γ is a max-co-clone if and only if it is a co-clone.

Proof: Lemma 20 implies that for any (n -ary) relation $R \in IL$ and any set $J = \{i_1, \dots, i_k\} \subseteq [n]$ the max-implementation $\exists_{\max}(x_{i_1}, \dots, x_{i_k})$ is equivalent to a sequence of ordinary existential quantifiers $\exists x_{i_1} \dots \exists x_{i_k}$. \square

6.3 Monotone relations

Recall that a relation is said to be monotone if it is invariant with respect to \wedge or \vee . In this section we consider relations invariant under \vee . A proof in the case of relations invariant under \wedge is similar. A monotone relation is called *nontrivial* if it does not belong to IR_2 .

Lemma 22 Let R be a nontrivial relation invariant under \vee . Then either $\text{IMP} \in \langle R \rangle_{\max}$, or $\text{OR} \in \langle R \rangle_{\max}$. In particular, if the all-zero tuple belongs to R then $\text{IMP} \in \langle R \rangle_{\max}$.

Proof: Observe that R is not self-complement, because as it follows from [28] (see also Fig. 1) all self complement monotone relations are trivial. Also if the all-one tuple does not belong to R , since R is invariant under \vee , some variables of R equal 0 in all tuples from R . Such variables can be quantified away, and the resulting relation is nontrivial as R is nontrivial. We may assume the all-one tuple is in R .

Suppose first that the all-zero tuple belongs to R . Therefore there is a tuple $\mathbf{a} \in R$ such that its complement does not belong to R . After a suitable rearrangement of variables $\mathbf{a} = (0, \dots, 0, 1, \dots, 1)$. Identify variables that take 1 in \mathbf{a} and also variables that take 0 in \mathbf{a} . The resulting relation is IMP .

Suppose now that the all-zero tuple does not belong to R . Then $\delta_1(x) = R(x, \dots, x)$. We also assume that R is a nontrivial relation of the minimal arity from $\langle R \rangle_{\max}$. Let x_1, \dots, x_n be the variables R depends on. We introduce a partial order on $[n]$ as follows: $i \leq_R j$ iff for any $\mathbf{a} \in R$ $\mathbf{a}[i] = 1$ implies $\mathbf{a}[j] = 1$. If $x_i \leq_R x_j$ for no $i, j \in [n]$, then for any $i \in [n]$ $R' = \exists_{\max} x_i (R(x_1, \dots, x_n) \wedge \delta_1(x_i))$ is a trivial relation, none of its projections equal $\{1\}$, and therefore the all-zero tuple belongs to R' . Hence $\mathbf{a}_{\{i\}} \in R$ where $\mathbf{a}_{\{i\}}[i] = 1$ and $\mathbf{a}_{\{i\}}[j] = 0$ for $j \neq i$. Since R is invariant under \vee , this implies that $R = \text{OR}^n$, and $\text{OR} \in \langle R \rangle_{\max}$ by Lemma 16(4).

Next, consider the case when $x_i \leq_R x_j$ for some $i, j \in [n]$. This means there are tuples $\mathbf{a}, \mathbf{b}, \mathbf{c} \in R$ such that $\mathbf{a}[i] = \mathbf{a}[j] = 0$ (since the projection of R on each variable is $\{0, 1\}$), $\mathbf{b}[i] = 0, \mathbf{b}[j] = 1$ (due to the minimality of R , there must be a tuple \mathbf{b} with $\mathbf{b}[i] \neq \mathbf{b}[j]$), and \mathbf{c} is the all-one tuple, in particular $\mathbf{c}[i] = \mathbf{c}[j] = 1$. Moreover, as R is invariant under \vee , we may assume that $\mathbf{b}[\ell] = 1$ whenever $\mathbf{a}[\ell] = 1$. After rearranging variables these tuples can be represented as follows

\mathbf{a}	$0 \dots 0$	$0 \dots 0$	$1 \dots 1$	$\in R$
\mathbf{b}	$0 \dots 0$	$1 \dots 1$	$1 \dots 1$	$\in R$
\mathbf{c}	$1 \dots 1$	$1 \dots 1$	$1 \dots 1$	$\in R$
	$x \dots x$	$y \dots y$	$z \dots z$	

Denote by R' the relation obtained from R by identifying variables as shown in the last row of the table. Relation R' contains tuples $\mathbf{a}' = (0, 0, 1), \mathbf{b}' = (0, 1, 1), \mathbf{c}' = (1, 1, 1)$. Observe that for no $\mathbf{d} \in R'$ we have $\mathbf{d}[1] = 1$ and $\mathbf{d}[2] = 0$. Therefore $\text{IMP}(x, y) = \exists_{\max} u (R'(x, y, u) \wedge \delta_1(u))$. \square

We first study max-co-clones not containing OR . By Lemma 16(1) and [14] (see also Table 1) $\langle \text{IMP} \rangle_{\max} = IM_2$.

Lemma 23 IM_2, IR_2, IR_0, IR_1 are max-co-clones.

Proof: Since IR_2, IR_0, IR_1 essentially contain only unary relations, the lemma for these co-clones is straightforward.

For IM_2 the result actually follows from Lemma 5 of [11]. However, as [11] uses a different framework, we give a short proof of this result here. Our proof can be derived from the one from [11]. Observe first that IMP satisfies the

property of log-supermodularity. A function $f: \{0, 1\}^n \rightarrow \mathbb{R}$ is said to be *log-supermodular* if for any \mathbf{a}, \mathbf{b}

$$f(\mathbf{a}) \cdot f(\mathbf{b}) \leq f(\mathbf{a} \vee \mathbf{b}) \cdot f(\mathbf{a} \wedge \mathbf{b}).$$

Here \wedge and \vee denote componentwise conjunction and disjunction. This definition can be extended to relations if they are treated as predicates, that is, functions with values 0, 1. As is easily seen, a relation is log-supermodular if and only if it is invariant under \wedge and \vee . First we show that if Γ is a set of log-supermodular relations then every relation from $\langle \Gamma \rangle_{\max}$ is log-supermodular. The property of log-supermodularity is obviously preserved by manipulations with variables and conjunction, because it is equivalent to the existence of certain polymorphisms. Suppose $R(x_1, \dots, x_n, y_1, \dots, y_m)$ is log-supermodular and $Q(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m) R(x_1, \dots, x_n, y_1, \dots, y_m)$. We associate every tuple $(\mathbf{a}, \mathbf{b}) \in \{0, 1\}^{n+m}$ with the set of ones in this tuple, and therefore can view R as a function on the power set of $[n + m]$. Take $\mathbf{a}, \mathbf{a}' \in \{0, 1\}^n$ and prove that $Q(\mathbf{a}) \cdot Q(\mathbf{a}') \leq Q(\mathbf{a} \vee \mathbf{a}') \cdot Q(\mathbf{a} \wedge \mathbf{a}')$. Let A be the set of tuples of the form $(\mathbf{a}, \mathbf{b}) \in \{0, 1\}^{n+m}$ and A' the set of tuples of the form $(\mathbf{a}', \mathbf{b}) \in \{0, 1\}^{n+m}$ viewed as subsets of $[n + m]$. Also, let $R(C) = \sum_{(\mathbf{c}, \mathbf{d}) \in C} R(\mathbf{c}, \mathbf{d})$ for $C \subseteq [n + m]$ and $f(x_1, \dots, x_n) = \sum_{y_1, \dots, y_m} R(x_1, \dots, x_n, y_1, \dots, y_m)$. Denote by $A \vee A'$ and $A \wedge A'$ the sets $A \vee A' = \{\mathbf{c} \vee \mathbf{c}' \mid \mathbf{c} \in A \text{ and } \mathbf{c}' \in A'\}$ and $A \wedge A' = \{\mathbf{c} \wedge \mathbf{c}' \mid \mathbf{c} \in A \text{ and } \mathbf{c}' \in A'\}$. Note that $f(\mathbf{a} \vee \mathbf{a}') = R(A \vee A')$ and $f(\mathbf{a} \wedge \mathbf{a}') = R(A \wedge A')$. Since R is log-supermodular, we know that $R(\mathbf{c}, \mathbf{d}) \cdot R(\mathbf{c}', \mathbf{d}') \leq R(\mathbf{c} \vee \mathbf{c}', \mathbf{d} \vee \mathbf{d}') \cdot R(\mathbf{c} \wedge \mathbf{c}', \mathbf{d} \wedge \mathbf{d}')$ for all $(\mathbf{c}, \mathbf{d}), (\mathbf{c}', \mathbf{d}') \in \{0, 1\}^{n+m}$. Thus, applying the Ahlswede-Daykin Four-Functions Theorem [1] with $\alpha = \beta = \gamma = \delta = R$,

$$f(\mathbf{a}) \cdot f(\mathbf{a}') = R(A) \cdot R(A') \leq R(A \vee A') \cdot R(A \wedge A') = f(\mathbf{a} \vee \mathbf{a}') \cdot f(\mathbf{a} \wedge \mathbf{a}'). \quad (1)$$

Now suppose $\mathbf{a}, \mathbf{a}' \in Q$. This means that $f(\mathbf{a}) = f(\mathbf{a}')$ and this number is the maximal number of extensions of a tuple from $\{0, 1\}^n$ to tuples from R . By (1) $f(\mathbf{a} \vee \mathbf{a}'), f(\mathbf{a} \wedge \mathbf{a}') \neq 0$ and either $f(\mathbf{a} \vee \mathbf{a}') \geq f(\mathbf{a})$ or $f(\mathbf{a} \wedge \mathbf{a}') \geq f(\mathbf{a}')$. However, as $f(\mathbf{a})$ is the maximal number of extensions, strict inequality is impossible, and we get $f(\mathbf{a} \vee \mathbf{a}') = f(\mathbf{a} \wedge \mathbf{a}') = f(\mathbf{a})$. Therefore $(\mathbf{a} \vee \mathbf{a}'), (\mathbf{a} \wedge \mathbf{a}') \in Q$, and so $Q(\mathbf{a}) \cdot Q(\mathbf{a}') \leq Q(\mathbf{a} \vee \mathbf{a}') \cdot Q(\mathbf{a} \wedge \mathbf{a}')$.

Thus $\langle IM_2 \rangle_{\max}$ contains only log-supermodular relations. However, as it was observed above, log-supermodularity of relations is equivalent to invariance under \wedge and \vee . Since, IM_2 is the class of all relations invariant under this two operations, we have $\langle IM_2 \rangle_{\max} = IM_2$. \square

Lemma 24 *Let $R \notin IM_2$. Then $\langle R, IMP \rangle_{\max} = II_2$.*

Proof: If R is not invariant under \vee and \wedge then the result follows by Lemma 19, since IMP is not affine or self-complement. Suppose R is invariant with respect \vee .

Recall that a relation $Q(x_1, \dots, x_n)$ is called *2-decomposable* if any tuple \mathbf{a} such that $(\mathbf{a}[i], \mathbf{a}[j]) \in \text{pr}_{\{i, j\}} Q$ for all $i, j \in [n]$ belongs to Q .

CASE 1. R is not 2-decomposable.

Let $I \subseteq [n]$ be a minimal set such that $\text{pr}_I R$ is not 2-decomposable, clearly, $|I| \geq 3$. Let $R' = \text{pr}_I R$. There is $\mathbf{a} \in \{0, 1\}^{|I|}$ such that for any $i \in I$ $\mathbf{a}_i \in R'$, where \mathbf{a}_i denotes the tuple such that $\mathbf{a}_i[i] \neq \mathbf{a}[i]$ and $\mathbf{a}_i[j] = \mathbf{a}[j]$ for $i \neq j$. Choose $i_1, i_2, i_3 \in I$, and set $I - \{i_1, i_2, i_3\} = \{i_4, \dots, i_k\}$ and

$$Q = \exists_{\max} x_{i_4} \dots \exists_{\max} x_{i_k} (R(x_1, \dots, x_n) \wedge \delta_{\mathbf{a}[i_4]}(x_{i_4}) \wedge \dots \wedge \delta_{\mathbf{a}[i_k]}(x_{i_k})).$$

As is easily seen, Q is not 2-decomposable, and moreover, $\text{pr}_{\{i_1, i_2, i_3\}} Q$ is not 2-decomposable. Let $Q' = \text{pr}_{\{i_1, i_2, i_3\}} Q$. There is $\mathbf{a} \in \{0, 1\}^3$ such that for any $i \in I$ $\mathbf{a}_i \in Q'$, where \mathbf{a}_i denotes the tuple such that $\mathbf{a}_i[i] \neq \mathbf{a}[i]$ and $\mathbf{a}_i[j] = \mathbf{a}[j]$ for $i \neq j$. Observe that there are at most one 1 among components of \mathbf{a} . Indeed, if, say, $\mathbf{a} = (1, 1, 0)$ then $\mathbf{a} = \mathbf{a}_1 \vee \mathbf{a}_2 \in Q'$. Suppose first that \mathbf{a} is the all-zero tuple. Then after rearranging variables these tuples can be

represented as follows

\mathbf{a}_1	1	0	0	0...0	0...0	0...0	1...1	0...0	1...1	1...1	1...1	$\in R$
\mathbf{a}_2	0	1	0	0...0	0...0	1...1	0...0	1...1	0...0	1...1	1...1	$\in R$
\mathbf{a}_3	0	0	1	0...0	1...1	0...0	0...0	1...1	1...1	0...0	1...1	$\in R$
\mathbf{a}	0	0	0	*	*	*	*	*	*	*	*	$\notin R$
	x	y	z	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	

Denote by Q'' the relation obtained from Q by identifying variables as shown in the last row of the table. Then set

$$S(x, y, z, t, u, v) = \exists_{\max} t_1 \exists_{\max} t_8 (Q''(x, y, z, t_1, z, y, x, t, u, v, t_8) \wedge \delta_0(t_1) \wedge \delta_1(t_8)).$$

Relation S contains tuples $\mathbf{b}_1 = (1, 0, 0, 1, 1, 0)$, $\mathbf{b}_2 = (0, 1, 0, 1, 0, 1)$, $\mathbf{b}_3 = (0, 0, 1, 0, 1, 1)$ but does not contain $(0, 0, 0, a, b, c)$ for any $a, b, c \in \{0, 1\}$. Next we set $S'(x, y, z) = \exists_{\max} t, u, v (S(x, y, z, t, u, v) \wedge \delta_1(t) \wedge \delta_1(u) \wedge \delta_1(v))$. Since S is invariant under \vee , it contains $\mathbf{b}_1 \vee \mathbf{b}_2$, $\mathbf{b}_2 \vee \mathbf{b}_3$, $\mathbf{b}_3 \vee \mathbf{b}_1$, and therefore S' contains tuples $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $(1, 1, 1)$, but does not contain $(0, 0, 0)$. Let also $S''(x, y, z) = S'(x, y, z) \wedge S'(z, x, y) \wedge S'(y, z, x)$. As is easily seen S'' is either OR^3 or $\{(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}$. In the former case we are done, while in the latter case we just observe that $\text{OR}(x, y) = \exists_{\max} z (S''(x, y, z) \wedge \delta_1(z))$.

Now suppose $\mathbf{a} = (0, 0, 1)$. As before we can construct a relation S such that $\mathbf{b}_1 = (0, 0, 0, 1, 1, 1)$, $\mathbf{b}_2 = (0, 1, 1, 0, 0, 1)$, $\mathbf{b}_3 = (1, 0, 1, 0, 1, 0)$ belong to S , but $(0, 0, 1, a, b, c)$ does not belong to S for any $a, b, c \in \{0, 1\}$. Since R is invariant under \vee tuples $\mathbf{b}_2 \vee \mathbf{b}_1$, $\mathbf{b}_3 \vee \mathbf{b}_1$, $\mathbf{b}_2 \vee \mathbf{b}_3 \vee \mathbf{b}_1$ also belong to S . Hence $(0, 0, 0, 1)$, $(0, 1, 1, 1)$, $(1, 0, 1, 1)$, $(1, 1, 1, 1) \in S'(x, y, z, t) = S(x, y, z, t, t, t)$, and $(0, 0, 1, 1) \notin S'$. Therefore $\text{OR}(x, y) = \exists_{\max} z \exists_{\max} t (S'(x, y, z, t) \wedge \delta_1(z) \wedge \delta_1(t))$.

CASE 2. R is 2-decomposable.

Since $\langle \text{IMP} \rangle_{\max}$ contains IM_2 and therefore all 2-decomposable relations whose binary projections are either trivial relations or IMP, relation R has to have a binary projection which is not one of them. As it and all its projections are invariant under \vee , the only nontrivial binary projections it may have are IMP and OR. Therefore for some $i, j \in [n]$ $\text{pr}_{\{i, j\}} R = \text{OR}$. There are $\mathbf{a}, \mathbf{b}, \mathbf{c} \in R$ such that $\mathbf{a}[i] = \mathbf{b}[j] = 0$ and $\mathbf{a}[j] = \mathbf{b}[i] = \mathbf{c}[i] = \mathbf{c}[j] = 1$, but for no $\mathbf{d} \in R$ $\mathbf{d}[i] = \mathbf{d}[j] = 0$. Note also that \mathbf{c} can be replaced with $\mathbf{c} \vee \mathbf{a} \vee \mathbf{b}$. After rearranging variables these tuples can be represented as follows

\mathbf{a}	0	1	0...0	0...0	0...0	1...1	1...1	$\in R$
\mathbf{b}	1	0	0...0	0...0	1...1	0...0	1...1	$\in R$
\mathbf{c}	1	1	0...0	1...1	1...1	1...1	1...1	$\in R$
\mathbf{d}	0	0	*	*	*	*	*	$\notin R$
	x	y	$z_1 \dots z_1$	$z_2 \dots z_2$	$z_3 \dots z_3$	$z_4 \dots z_4$	$z_5 \dots z_5$	

Denote by R' the relation obtained from R by identifying variables as shown in the last row of the table. Then set

$$Q(x, y, z) = \exists_{\max} z_1 \exists_{\max} z_5 (Q(x, y, z_1, z, x, y, z_5) \wedge \delta_0(z_1) \wedge \delta_1(z_5)).$$

Relation Q contains tuples $(0, 1, 0)$, $(1, 0, 0)$, $(1, 1, 1)$, and $(1, 1, 0)$, as it is invariant under \vee , but does not contain $(0, 0, a)$ for any $a \in \{0, 1\}$. Then $\text{OR}(x, y) = \exists_{\max} z (Q(x, y, z) \wedge \delta_0(z))$. \square

Next we consider max-co-clones containing OR, but not IMP.

Let $R(x_1, \dots, x_n)$ be a relation. If $i, j \in [n]$ are such that $\mathbf{a}[i] = \mathbf{a}[j]$ for any $\mathbf{a} \in R$, we write $i \sim_R j$. Clearly, \sim_R is an equivalence relation on $[n]$; its class containing i will be denoted by $S_R(i)$ or $S_R(x_i)$. Let also O_R denote the set of variables x_j such that there is $\mathbf{b} \in R$ with $\mathbf{b}[j] = 1$. An n -tuple \mathbf{a} is said to be \sim_R -conforming if (a) $\mathbf{a}[i] = \mathbf{a}[j]$ whenever $i \sim_R j$, and (b) $\mathbf{a}[i] = 0$ whenever $i \notin O_R$. When considered ordered with respect to the

natural component-wise order ($0 \leq 1$), \sim_R -conforming tuples form a poset isomorphic to $\{0, 1\}^{k_R}$, where k_R is the number of \sim_R -classes except for the class $[n] - O_R$. In what follows \leq and $<$ will denote relations on the set of \sim_R -conforming tuples for appropriate R . We say that a relation $R(x_1, \dots, x_n)$ satisfies the *filter property* if for any $\mathbf{a} \in R$ any \sim_R -conforming tuple \mathbf{a}' with $\mathbf{a} \leq \mathbf{a}'$ belongs to R . The filter property implies that if R is considered as a subset of the ordered set $\{0, 1\}^{k_R}$, then it is an order filter in this set. In particular, it is completely determined by its minimal (with respect to \leq) elements, or equivalently by the maximal elements not belonging to R . We say that R satisfies the *r-filter property*, if it satisfies the filter property, and every maximal tuple not belonging to R contains zeros in at most r classes of \sim_R from O_R .

Lemma 25 (1) A relation R belongs to IS_{12} if and only if it satisfies the filter property.
(2) A relation R belongs to IS_{12}^r if and only if it satisfies the *r-filter property*.

Proof: (1) Suppose $R(x_1, \dots, x_n) \in IS_{12}$. Then by Proposition 3 of [14] the set EQ, δ_0, δ_1 and OR^m , $m \geq 2$ is a plain basis of IS_{12} , and therefore R can be represented by a conjunctive formula Φ containing variables x_1, \dots, x_n , relations EQ, δ_0, δ_1 , and OR^m . Let $\mathbf{a} \in R$, and let \mathbf{b} be a \sim_R -conforming tuple such that $\mathbf{a} \leq \mathbf{b}$. We show that it belongs to R . Clearly, \mathbf{b} satisfies all the δ_1 relations. Also, it satisfies all the δ_0 relations, if $\delta_0(x_j)$ belongs to Φ then $j \notin O_R$ and $\mathbf{b}[j] = 0$. Since \mathbf{b} contains 0 only in the positions \mathbf{a} does, every relation OR^m is satisfied by \mathbf{b} . Finally, if $\text{EQ}(x_{j_1}, x_{j_2})$ belongs to Φ , then $j_1 \sim_R j_2$, therefore all the EQ relations remain satisfied by \mathbf{b} .

Suppose now that $R(x_1, \dots, x_n)$ satisfies the filter property. Let $W, Z \subseteq [n]$ be the sets of variables such that for all $\mathbf{a} \in R$ $\mathbf{a}[i] = 1$ (respectively, $\mathbf{a}[i] = 0$) for $i \in W$ ($i \in Z$). Let also $\mathbf{a}_1, \dots, \mathbf{a}_\ell$ be the maximal tuples not from R . By Z_j we denote the set of $i \in O_R$ such that $\mathbf{a}_j[i] = 0$. Suppose Z_j contains elements from m_j classes of \sim_R . We construct a formula Φ using variables x_1, \dots, x_n and relations EQ, δ_0, δ_1, OR^m , and prove that it represents R . Formula Φ includes

- (1) $\delta_0(x_i)$ for each $i \in Z$ and $\delta_1(x_i)$ for each $i \in W$;
- (2) $\text{EQ}(x_i, x_j)$ for any pair $x_i, x_j, i \sim_R j$;
- (3) $OR^{m_j}(x_{i_1}, \dots, x_{i_{m_j}})$ for any $\mathbf{a}_j, j \in [\ell]$, and any i_1, \dots, i_{m_j} such that i_1, \dots, i_{m_j} belong to different \sim_R -classes from Z_j .

Let the resulting relation be denoted by Q . By what is proved above Q satisfies the filter property. It is straightforward that $O_Q = O_R$ and the maximal tuples not in Q are the same as those of R . Therefore $Q = R$.

(2) Suppose first that R satisfies the *r-filter property*. Then it can be represented by a formula Φ as in part (1) and for every relation OR^m used $m \leq r$. Therefore $R \in IS_{12}^r$.

Let now $R(x_1, \dots, x_n) \in IS_{12}^r$, and therefore can be represented by a formula Φ in x_1, \dots, x_n , and relations EQ, δ_0, δ_1 , and OR^m for $m \leq r$. We need to study the structure of maximal tuples from the complement of R . We use the notation from part (1). Let \mathbf{a} be such a tuple. It is \sim_R -conforming, so, $\mathbf{a}[i] = 0$ for all $i \in Z$, and $\mathbf{a}[i] = \mathbf{a}[j]$ for any $i \sim_R j$. This means that \mathbf{a} satisfies all the δ_0 and EQ relations in Φ . If \mathbf{a} violates a relation δ_1 and there is $i \notin W$ such that $\mathbf{a}[i] = 0$ then \mathbf{a} is not maximal in the complement of R . Therefore $\mathbf{a}[i] = 0$ if and only if $i \in W$, and W is a single \sim_R -class. Suppose \mathbf{a} violates a relation $OR^m(x_{i_1}, \dots, x_{i_m})$, and let $D = S(i_1) \cup \dots \cup S(i_m)$. If there is $i \in O_R - D$ such that $\mathbf{a}[i] = 0$ then the tuple \mathbf{b} given by $\mathbf{b}[j] = 1$ if $j \in S(i)$ and $\mathbf{b}[j] = \mathbf{a}[j]$ otherwise does not belong to R and $\mathbf{a} < \mathbf{b}$, a contradiction. Therefore the set of zeros of any maximal tuple from the complement of R spans at most r classes of \sim_R , as required. \square

Let Γ be a max-co-clone of monotone relations. By $\text{or}(\Gamma)$ we denote the maximal m such that $OR^m \in \langle \Gamma \rangle_{\max}$. If a maximal number m does not exist we set $\text{or}(\Gamma) = \infty$.

Lemma 26 For any set $\Gamma \subseteq IS_{12}$ of monotone relations

$$\langle \Gamma \rangle_{\max} = \langle \{OR^m \mid m \leq \text{or}(\Gamma)\} \rangle_{\max} \quad \text{or} \quad \langle \Gamma \rangle_{\max} = \langle \{OR^m \mid m \leq \text{or}(\Gamma)\} \rangle_{\max} \cup \{\delta_0\}.$$

Proof: It suffices to show that if Γ contains a relation R with a maximal tuple that spans k classes of \sim_R , then $\text{OR}^k \in \langle \Gamma \rangle_{\max}$. Let R be such a relation. Applying \exists_{\max} we may assume that the sets W and Z for R are empty; applying identification of variables we may assume that every set $S(i)$ is a singleton. Now let \mathbf{a} be a maximal tuple that spans k classes of \sim_R , and I the set of positions such that $\mathbf{a}[i] = 0$ if and only if $i \in I$; without loss of generality assume $I = [k]$. Since R satisfies the filter property, for any $(b_1, \dots, b_k) \in \text{pr}_{[k]} R$ the tuple $(b_1, \dots, b_k, 1, \dots, 1)$ belongs to R . Observe that identifying all the variables of R we make sure that $\delta_1 \in \langle \Gamma \rangle_{\max}$. Therefore the relation given by

$$Q(x_1, \dots, x_k) = \exists_{\max}(x_{k+1}, \dots, x_n)(R(x_1, \dots, x_n) \wedge \delta_1(x_{k+1}) \wedge \dots \wedge \delta_1(x_n))$$

belongs to $\langle \Gamma \rangle_{\max}$. It remains to show that $Q = \text{OR}^k$. By the filter property of R for any b_1, \dots, b_k that are not all zeros $(b_1, \dots, b_k, 1, \dots, 1) \in R$. Therefore $(b_1, \dots, b_k) \in Q$. On the other hand, $(0, \dots, 0, 1, \dots, 1) \notin R$.

It remains to show that for any $R(x_1, \dots, x_n) \in IS_{12}$ such that $\mathbf{a}_{[n]} \notin R$ (the all-ones tuple), $\delta_0 \in \langle R \rangle_{\max}$. By the filter property of R if $\mathbf{a}_{[n]} \notin R$ there is $i \in [n]$ such that $\mathbf{a}[i] = 0$ for all $\mathbf{a} \in R$. Let $I \subseteq [n]$ be the set of all such coordinate positions; without loss of generality we may assume that $I = [m]$. Since $\delta_1 \in \langle R \rangle_{\max}$, we have

$$\delta_0(x) = \exists_{\max} y(R(x, \dots, x, y, \dots, y) \wedge \delta_1(y)),$$

where x is in the first m positions. □

Lemma 27 *Every co-clone $IS_1, IS_{12}, IS_1^r, IS_{12}^r$ for $r \in \{2, 3, \dots\}$ is a max-co-clone.*

Proof: First we show that every IS_{12}, IS_{12}^r is a max-co-clone. By Lemma 25 it suffices to prove that if every relation from Γ satisfies the filter or r -filter property, then so does every relation from $\langle \Gamma \rangle_{\max}$. These properties are preserved by manipulations with variables and conjunction, because IS_{12}, IS_{12}^r are co-clones. It remains to show that they are also preserved by max-implementation.

Suppose $R(x_1, \dots, x_n, y_1, \dots, y_m)$ satisfies the filter property and $Q(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m) R(x_1, \dots, x_n, y_1, \dots, y_m)$. Observe that we may assume that for any x_i the set $S(x_i)$ does not contain any variable y_j . Indeed, if $\mathbf{a}[i] = \mathbf{b}[j]$ for any assignment (\mathbf{a}, \mathbf{b}) that satisfies R , then we can identify these two variables, and denote the new variable by x_i . The number of extensions of any assignment to x_1, \dots, x_n does not change, therefore the relation Q defined in the same way from the new relation does not change.

Choose a representation Φ of Q that uses $\text{OR}^r, \text{EQ}, \delta_0, \delta_1$. Such a representation exists as the listed relations constitute a plain basis for IS_{12} by [14] (see Table 1). Take $\mathbf{a} \in Q$ and $x_i \in O_Q$; let \mathbf{a}' be the tuple such that $\mathbf{a} \leq \mathbf{a}'$. It suffices to verify that every extension \mathbf{b} of \mathbf{a} is also extension of \mathbf{a}' . Indeed, if this is the case, since \mathbf{a} has the maximum number of extensions, so does \mathbf{a}' , and thus $\mathbf{a}' \in Q$. Suppose $(\mathbf{a}, \mathbf{b}) \in R$. Then $(\mathbf{a}', \mathbf{b})$ satisfies every relation OR^r from Φ , as this tuple contains 1 in every position (\mathbf{a}, \mathbf{b}) does. It also satisfies every relation EQ , because there is no relation of the form $\text{EQ}(x_\ell, y_j)$, and $\mathbf{a}'[i] = \mathbf{a}'[j]$ whenever $i \sim_R j$. Finally, δ_0 and δ_1 are also satisfied, because no value is changed in the scopes of the former, and no value is changed to 0 in the scope of the latter.

Next we prove that the number of \sim_R -classes spanned by zeros of maximal tuples from the complement of Q does not exceed that of R . More precisely we show that (1) $S_R(x_i) \cap \{x_1, \dots, x_n\} \subseteq S_Q(x_i)$ for any $i \in [n]$, and (2) for every maximal tuple $\mathbf{a} \notin Q$ there is $\mathbf{b} \in \{0, 1\}^m$ such that (\mathbf{a}, \mathbf{b}) is a maximal tuple not belonging to R .

The first claim is obvious, as $Q \subseteq \text{pr}_{[n]} R$ and therefore if $\mathbf{a}[i] = \mathbf{a}[j]$ for any $(\mathbf{a}, \mathbf{b}) \in R$ then $\mathbf{c}[i] = \mathbf{c}[j]$ for any $\mathbf{c} \in Q$. Observe that we may assume that $\text{pr}_j R = \{0, 1\}$ for any $j \in \{n+1, \dots, n+m\}$, since otherwise such a variable does not affect the number of extensions of tuples from $\text{pr}_{[n]} R$. For the second claim let \mathbf{a} be a maximal tuple not belonging to Q . Suppose first that $\mathbf{a} \notin \text{pr}_{[n]} R$. Since for any $\mathbf{a}' \in \text{pr}_{[n]} R$ the tuple $(\mathbf{a}', 1, \dots, 1)$ belongs to R , the tuple $(\mathbf{a}, 1, \dots, 1)$ is a maximal tuple not belonging to R . Next assume $\mathbf{a} \in \text{pr}_{[n]} R$. Let $E(\mathbf{c})$ denote the set of extensions of a tuple $\mathbf{c} \in \text{pr}_{[n]} R$ to a tuple from R . Due to the filter property of R and the assumption that no set $S(x_i)$ contains any y_j , if $\mathbf{c} \leq \mathbf{c}'$ then $E(\mathbf{c}) \subseteq E(\mathbf{c}')$. As \mathbf{a} is a maximal tuple not belonging to Q , the number of extensions of any tuple \mathbf{a}' , $\mathbf{a} < \mathbf{a}'$, is the same, including the all-one tuple $\mathbf{a}_{[n]}$. However, for any such tuple \mathbf{a}' , $E(\mathbf{a}') \subseteq E(\mathbf{a}_{[n]})$

and yet $|E(\mathbf{a}')| = |E(\mathbf{a}_{[n]})|$ implying $E(\mathbf{a}') = E(\mathbf{a}_{[n]})$. Since $|E(\mathbf{a})| < |E(\mathbf{a}')|$ for any tuple \mathbf{a}' , $\mathbf{a} < \mathbf{a}'$, there is \mathbf{b} such that $(\mathbf{a}, \mathbf{b}) \notin R$ and $(\mathbf{a}', \mathbf{b}) \in R$ for any tuple \mathbf{a}' , $\mathbf{a} < \mathbf{a}'$. Choose a maximal \mathbf{b}' , $\mathbf{b} \leq \mathbf{b}'$, with this property. We need to show that $(\mathbf{a}, \mathbf{b}')$ is a maximal tuple not belonging to R . For any $\mathbf{b}'' > \mathbf{b}'$ the tuple $(\mathbf{a}, \mathbf{b}'') \in R$, because, by the choice of \mathbf{b}' , it is a maximal tuple such that $(\mathbf{a}, \mathbf{b}') \notin R$. For any \mathbf{a}' , $\mathbf{a} < \mathbf{a}'$, the tuple $(\mathbf{a}', \mathbf{b})$ belongs to R , and therefore $(\mathbf{a}', \mathbf{b}') \in R$.

Next we show that $\langle IS_1^r \rangle_{\max} = IS_1^r$. Co-clone IS_1^r contains all relations from IS_{12}^r invariant under the constant function 1. So, we prove that any relation $R \in \langle IS_1 \rangle_{\max}$ contains the all-one tuple. Relations EQ, δ_1 , and OR^r satisfy this condition. Manipulations with variables and conjunction preserves this property. It remains to verify that \exists_{\max} also preserves this property in IS_{12} . Let $R(x_1, \dots, x_n, y_1, \dots, y_m) \in IS_{12}$ and $(1, \dots, 1, 1, \dots, 1) \in R$. Let also $Q(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m)R(x_1, \dots, x_n, y_1, \dots, y_m)$. As before we may assume that for any x_i the set $S(x_i)$ does not contain any variable y_j . Then since $E(\mathbf{a}) \subseteq E(\mathbf{a}_{[n]})$, where $\mathbf{a}_{[n]}$ is the all-one tuple, for any $\mathbf{a} \in \text{pr}_{[n]}R$, $\mathbf{a}_{[n]} \in Q$. \square

Lemma 28 *Let $R \notin IS_{12}$, then $\langle R, OR \rangle_{\max} = II_2$.*

Proof: First of all R can be assumed to be closed under \vee . Indeed, OR is not self-complement, affine, or closed under \wedge ; so if R is not closed under \vee the result follows from Lemma 19. We also may assume that every unary projection of R contains two elements. Next, observe that we can also assume that for each variable x of R the set $S(x)$ contains only one element. Indeed, construct a relation R' by identifying all variables in every set of the form $S(x)$. It now suffices to verify that $R' \notin IS_{12}$ whenever $R \notin IS_{12}$. To see this note that R can be obtained from R' through adding new variables and imposing equality relations.

If R contains the all-zero tuple then by Lemma 22 $\text{IMP} \in \langle R \rangle_{\max}$ and the result follows from Lemma 17.

Suppose that the all-zero tuple does not belong to R . We show that either R satisfies the filter property, and therefore belongs to IS_{12} , or there is a nontrivial relation $Q \in \langle R \rangle_{\max}$ containing the all-zero tuple. By what is proved above it implies the result.

For $\mathbf{a} \in R$ we denote by $R_{\mathbf{a}}$ the relation obtained as follows. Let $O(\mathbf{a})$ denote the set of coordinate positions in which \mathbf{a} equals 1. Then

$$R_{\mathbf{a}} = \exists_{\max}(x_i)_{i \in O(\mathbf{a})} (R(x_1, \dots, x_n \wedge \bigwedge_{i \in O(\mathbf{a})} \delta_1(x_i)).$$

If $R_{\mathbf{a}}$ is a nontrivial relation then we are done, since the all-zero tuple belongs to $R_{\mathbf{a}}$. Therefore assume that every relation $R_{\mathbf{a}}$ is trivial. Observe that since $\mathbf{a} \vee \mathbf{b} \in R$ for any $\mathbf{b} \in R$ and $\text{pr}_{[n]-O(\mathbf{a})}(\mathbf{a} \vee \mathbf{b}) = \text{pr}_{[n]-O(\mathbf{a})}\mathbf{b}$, we have $R_{\mathbf{a}} = \text{pr}_{[n]-O(\mathbf{a})}R$. Therefore every set of the form $S(x)$ for $R_{\mathbf{a}}$ is 1-element. Hence $R_{\mathbf{a}} = \{0, 1\}^{n-|O(\mathbf{a})|}$. In particular, for any $\mathbf{a} \in R$ and any $i \notin O(\mathbf{a})$ the tuple \mathbf{b} obtained from \mathbf{a} by changing $\mathbf{a}[i]$ to 1 belongs to R . Thus R satisfies the filter property. \square

Proposition 29 *Every max-co-clone of monotone relations containing a nontrivial relation equals one of IS_1 , IS_{12} , IS_1^i , IS_{12}^i for $i \in \{2, 3, \dots\}$, IM_2 .*

Proof: By Lemmas 23 and 27 all these sets are max-co-clones. By Lemma 24 and the observation that $\langle \text{IMP} \rangle_{\max} = IM_2$, max-co-clone IM_2 is the only max-co-clone containing IMP. By Lemma 28 IS_{12} is the greatest max-co-clone containing OR. Thus it remains to prove that there are no max-co-clones containing OR and different from $IS_1, IS_{12}, IS_1^i, IS_{12}^i$ for $i \in \{2, 3, \dots\}$. It follows from Lemma 26. \square

6.4 Self-complement max-co-clones

In this section we consider the remaining case of self-complement max-co-clones.

Proposition 30 *There is only one max-co-clone of self-complement relations that is not a subclone of IL_2 . It is IN_2 , the clone of all self-complement relations.*

The proposition follows from the following four lemmas.

Lemma 31 IN_2 is a max-co-clone.

Proof: We need to prove that IN_2 is closed under manipulations with variables, conjunction, and max-implementation. Since IN_2 is a co-clone, it is closed under the first two operations. Let $R(x_1, \dots, x_n, y_1, \dots, y_m) \in IN_2$ and $Q(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m)R(x_1, \dots, x_n, y_1, \dots, y_m)$. Let $\mathbf{a} \in Q$ and let $\neg \mathbf{a}$ denote its complement. Then for each extension $(\mathbf{a}, \mathbf{c}) \in R$ of \mathbf{a} the tuple $(\neg \mathbf{a}, \neg \mathbf{c})$ belongs to R , as R is self-complement, and $(\neg \mathbf{a}, \neg \mathbf{c})$ is an extension of $\neg \mathbf{a}$. Therefore $\neg \mathbf{a}$ has the same number of extensions as \mathbf{a} , and so $\neg \mathbf{a} \in Q$. Thus, Q is self-complement. \square

Lemma 32 *Let R be a self complement relation that does not belong to IL_2 (that is, non-affine), then $\text{Compl}_{3,0} \in \langle R \rangle_{\max}$ or $\text{Compl}_{1,2} \in \langle R \rangle_{\max}$.*

Proof: Let $R(x_1, \dots, x_n)$ satisfy the conditions of the lemma. There are two cases.

CASE 1. R does not contain the all-zero tuple.

Observe first that in this case $\langle R \rangle_{\max}$ contains the disequality relation. Indeed, let $\mathbf{a} \in R$ and let $I \subseteq [n]$ be the set of indices such that $\mathbf{a}[i] = 0$ if and only if $i \in I$. Since the all-zero tuple does not belong to R , $I \neq [n]$. Without loss of generality let $I = [m]$. Then it is easy to see that

$$R(\underbrace{x, \dots, x}_{m \text{ times}}, y, \dots, y)$$

is the disequality relation.

As $R \notin IL_2$, by Lemma 4.10 of [15] there are tuples $\mathbf{a}, \mathbf{b}, \mathbf{c} \in R$ such that $\mathbf{d} = \mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \notin R$. Rearranging the variables these tuples can be represented as shown in the table below.

\mathbf{a}	0...0	0...0	0...0	0...0	1...1	1...1	1...1	1...1	$\in R$
\mathbf{b}	0...0	0...0	1...1	1...1	0...0	0...0	1...1	1...1	$\in R$
\mathbf{c}	0...0	1...1	0...0	1...1	0...0	1...1	0...0	1...1	$\in R$
\mathbf{d}	0...0	1...1	1...1	0...0	1...1	0...0	0...0	1...1	$\notin R$
	$x \dots x$	$y \dots y$	$z \dots z$	$s \dots s$	$t \dots t$	$u \dots u$	$v \dots v$	$w \dots w$	

Denote by R' the relation obtained from R by identifying variables as shown in the last row of the table, and then set

$$Q(x, y, z, t) = \exists_{\max} s \exists_{\max} u \exists_{\max} v \exists_{\max} w (R'(x, y, z, s, t, u, v, w) \wedge \text{NEQ}(x, w) \wedge \text{NEQ}(y, v) \wedge \text{NEQ}(z, u) \wedge \text{NEQ}(t, s)).$$

Relation R'' contains tuples $(0, 0, 0, 1)$, $(0, 0, 1, 0)$, $(0, 1, 0, 0)$ but does not contain $(0, 1, 1, 1)$, and so does not belong to IL_2 .

There are 16 cases depending on whether or not tuples (a) $(0, 0, 1, 1)$, (b) $(0, 1, 0, 1)$, (c) $(0, 1, 1, 0)$, and (d) $(0, 0, 0, 0)$ belong to R'' (remember, this relation is self complement). If none of them belong to R'' then $\text{Compl}_{3,0}(x, y, z) = \exists_{\max} t R''(t, x, y, z)$. Suppose first $(0, 0, 0, 0) \notin R''$. If (a) belongs to R'' then $\text{Compl}_{3,0}(x, y, z) = R''(x, x, y, z)$; if (b) is in R'' then $\text{Compl}_{3,0}(x, y, z) = R''(x, y, x, z)$; finally, if (c) is in R'' then $\text{Compl}_{3,0}(x, y, z) = R''(x, y, z, x)$. Suppose now (d) belongs to R . If (a) is not there then $\text{Compl}_{1,2}(x, y, z) = R''(x, x, y, z)$. If (a) is also in R , then $\text{Compl}_{1,2}(x, y, z) = R''(x, y, z, z)$.

CASE 2. The all-zero tuple belongs to R .

Again by Lemma 4.10 of [15] there are tuples $\mathbf{a}, \mathbf{b}, \mathbf{c} \in R$ such that $\mathbf{d} = \mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \notin R$, but \mathbf{a} can be chosen to be the all-zero tuple. Then after rearranging variables these tuples can be represented as follows

\mathbf{a}	$0 \dots 0$	$0 \dots 0$	$0 \dots 0$	$0 \dots 0$	$\in R$
\mathbf{b}	$0 \dots 0$	$0 \dots 0$	$1 \dots 1$	$1 \dots 1$	$\in R$
\mathbf{c}	$0 \dots 0$	$1 \dots 1$	$0 \dots 0$	$1 \dots 1$	$\in R$
\mathbf{d}	$0 \dots 0$	$1 \dots 1$	$1 \dots 1$	$0 \dots 0$	$\notin R$
	$x \dots x$	$y \dots y$	$z \dots z$	$t \dots t$	

Denote by R' the relation obtained from R by identifying variables as shown in the last row of the table. Relation R' contains tuples $(0, 0, 0, 0)$, $(0, 0, 1, 1)$, $(0, 1, 0, 1)$ but does not contain $(0, 1, 1, 0)$, and so does not belong to IL_2 .

There are 16 cases depending on whether or not tuples (a) $(0, 0, 0, 1)$, (b) $(0, 0, 1, 0)$, (c) $(0, 1, 0, 0)$, and (d) $(1, 0, 0, 0)$ belong to R' . If none of the tuples belong to R' or all of them belong to R' , then $\text{Compl}_{2,1}(x, y, z) = \exists_{\max} t R'(t, x, y, z)$. In the first case it is 1-quantification, and in the second case it is 2-quantification. If exactly one of (a) and (b) belongs to R' then up to permutation of variables $\text{Compl}_{1,2}(x, y, z) = R'(x, x, y, z)$. If exactly one of (a) and (d) belongs to R' then up to permutation of variables $\text{Compl}_{1,2}(x, y, z) = R'(x, y, y, z)$. Finally, if exactly one of (c) and (d) belongs to R' then up to permutation of variables $\text{Compl}_{1,2}(x, y, z) = R'(x, y, z, z)$. \square

Lemma 33 *If $k + \ell \geq 3$ then $\langle \text{Compl}_{k,\ell} \rangle_{\max} = IN_2$.*

Proof: Observe first that

$$\begin{aligned}
\text{Compl}_{k,\ell}(x_1, \dots, x_{k+\ell}) &= \exists_{\max} y \text{Compl}_{k,\ell+1}(x_1, \dots, x_{k+\ell}, y), \\
\text{Compl}_{k,\ell}(x_1, \dots, x_{k+\ell}) &= \exists_{\max} y (\text{Compl}_{k+1,\ell-1}(x_1, \dots, x_k, y, x_{k+2}, x_{k+\ell}) \\
&\quad \wedge \text{NEQ}(y, x_{k+1})), \quad \text{and} \\
\text{Compl}_{k,0}(x_1, \dots, x_k) &= \exists_{\max} y \text{Compl}_{k+1,0}(x_1, \dots, x_k, y).
\end{aligned} \tag{2}$$

Also,

$$\begin{aligned}
&\text{Compl}_{k,\ell}(x_1, \dots, x_{k+\ell}) \\
&= \exists_{\max} y_1, \dots, y_k \text{Compl}_{k+\ell,0}(y_1, \dots, y_k, x_{k+1}, \dots, x_{k+\ell+1}) \wedge \text{NEQ}(y_1, x_1) \wedge \dots \wedge \text{NEQ}(y_k, x_k).
\end{aligned}$$

Since $\text{NEQ} = \text{Compl}_{2,0}$, the equalities above imply that if $k' + \ell' \leq k + \ell$ then $\text{Compl}_{k',\ell'} \in \langle \text{Compl}_{k,\ell} \rangle_{\max}$.

Now it suffices to show that $\text{Compl}_{2k,0} \in \langle \text{Compl}_{k+1,0} \rangle_{\max}$. We start with the relation given by the following formula

$$\begin{aligned}
\Phi(x_1, \dots, x_{2k}, y_1, \dots, y_{\binom{k}{2}}) &= \bigwedge_{I=\{i_1, \dots, i_k\} \subseteq [2k]} \text{Compl}_{k+1,0}(x_{i_1}, \dots, x_{i_k}, y_{j_I}) \\
&\quad \wedge \bigwedge_{I \subseteq [2k], |I|=k} \text{NEQ}(y_{j_I}, y_{j_{I^c}}).
\end{aligned}$$

Here j_I is some enumeration of the k -element subsets of $[2k]$. We are interested in assignments of x_1, \dots, x_{2k} and the number of ways such an assignment can be extended to a satisfying assignment of Φ . First, observe that the only assignments of x_1, \dots, x_{2k} that can not be extended are the all-zero and all-one assignment. Second, since Φ is symmetric with respect of permutations of $\{x_1, \dots, x_{2k}\}$ in the sense that for any permutation of this set there is a permutation of the y_i 's that keeps the formula unchanged, the number of extensions of an assignment of x_1, \dots, x_{2k} depends only on the number of 0's in the assignment. We will denote this number by $N_\Phi(m)$, where m is the number of zeros. Notice that Φ defines a self-complement relation, therefore, we always assume that the number of zeros is at

least k . As is easily seen, if a tuple \mathbf{a} has $m \geq k$ zeros, it can be extended in $N_\Phi(m) = 2^{\frac{1}{2}\binom{k}{2k} - \binom{k}{m}}$ ways. Indeed, y_I is uniquely defined by \mathbf{a} if I or \bar{I} is a subset of the set of zeros of \mathbf{a} . Otherwise it can take any value independently of the values of other variables, except that $y_{j_I} \neq y_{j_{\bar{I}}}$.

Let $Q(x_1, \dots, x_k, y)$ be the relation given by: if $x_1 = \dots = x_k$ then y can be any, otherwise $y = x_1$. Relation Q is an intersection of some relations $\text{Compl}_{k', \ell'}$ with $k' + \ell' = k + 1$. Therefore by (2) it belongs to $\langle \text{Compl}_{k+1, 0} \rangle_{\max}$. Set

$$\Phi'(x_1, \dots, x_{2k}, y_1, \dots, y_{\binom{k}{2k}}) = \bigwedge_{I=\{i_1, \dots, i_k\} \subseteq [2k]} Q(x_{i_1}, \dots, x_{i_k}, y_{j_I}),$$

and consider $\Psi = \Phi \wedge \Phi'$, where Φ, Φ' have the same variables x_i , but the sets of the auxiliary variables y_i are disjoint. Observe that $N_\Psi(m) = N_\Phi(m) \cdot N_{\Phi'}(m)$. Similarly to Φ , $N_{\Phi'}(m) = 2^{\binom{k}{m}}$, provided $m \geq k$. Indeed, variable y_{j_I} can be assigned any value if $x_i = 0$ for all $i \in I$; otherwise y_{j_I} can take only one value. Therefore for any $m \neq 0$

$$N_\Psi(m) = 2^{\frac{1}{2}\binom{k}{2k} - \binom{k}{m}} \cdot 2^{\binom{k}{m}} = 2^{\frac{1}{2}\binom{k}{2k}}$$

and $N_\Psi(0) = 0$. Thus $\text{Compl}_{2k, 0} = \exists_{\max}(y_1, \dots, y_{\binom{k}{2k}})\Psi$.

It now remains to apply Proposition 3 of [14] that claims, in particular, that the relation $\text{Compl}_{k, \ell}$ constitute a plain basis of IN_2 . \square

7 CONCLUSION

The results of the previous section can be used to reprove some complexity results, namely, that of [18]. If for counting problems A and B there are approximation preserving reductions from A to B , and from B to A , we denote it by $A =_{AP} B$. The problem $\#CSP(\text{IMP})$ plays a special role in this result. This problem can also be interpreted as the problem of counting the number of independent sets in a bipartite graph, $\#BIS$, or as the problem of counting antichains in a partially ordered set [17]. The problem of counting the number of satisfying assignments to a CNF, $\#SAT$, is predictably the most difficult problem among counting CSPs.

Theorem 34 *Let Γ be a set of relations over $\{0, 1\}$. If every relation in Γ is affine then $\#CSP(\Gamma)$ is in solvable in polynomial time. Otherwise if every relation in Γ is in IM_2 then $\#CSP(\Gamma) =_{AP} \#BIS$. Otherwise $\#CSP(\Gamma) =_{AP} \#SAT$.*

Proof: The $\#CSP$ over affine relations can be solved exactly in polynomial time, as it is proved in [15]. If Γ contains OR or NAND, the problem $\#CSP(\Gamma)$ is interreducible with $\#SAT$ by Theorem 3 of [17] (observe that the problem $\#IS$ of counting the number of independent sets in a graph can be represented as $\#CSP(\text{NAND})$). By Theorems 3 and 15 this leaves only two max-co-clones to consider, IM_2 and IN_2 . Since IM_2 is generated by IMP and by Lemma 22, for any $\Gamma \subseteq IM_2$ the problem $\#CSP(\Gamma)$ is either polynomial time solvable, or is interreducible with $\#BIS$. The remaining max-co-clone, IN_2 is generated by $\text{Compl}_{3, 0}$ that contains all tuples such that not all their entries are equal; this is why it is sometimes called the Not-All-Equal relation, or NAE. Therefore for any $\Gamma \subseteq IN_2$ such that $\Gamma \not\subseteq IL_3$ the problem $\#CSP(\Gamma)$ is interreducible with $\#CSP(\text{NAE})$. By [30] the decision problem $\text{CSP}(\text{NAE})$ is NP-complete. Therefore by Theorem 1 of [17] $\#CSP(\text{NAE})$ is interreducible with $\#SAT$. \square

Observe also that some co-clones are not max-co-clones, even those co-clones are generated (or ‘determined’) by surjective functions. For instance, IS_{00} or IS_{01} . Since on a 2-element set every quantification with \exists_{\max}^1 is equivalent to either existential, or universal quantification, and therefore $\langle \Gamma \rangle_{\max}^1$ can be any set of relations of the form $\text{Inv}(C)$ for a set of surjective functions C , we obtain the following

Corollary 35 *There is a set Γ of relations on $\{0, 1\}$ such that $\langle \Gamma \rangle_{\max} \neq \langle \Gamma \rangle_{\max}^1$.*

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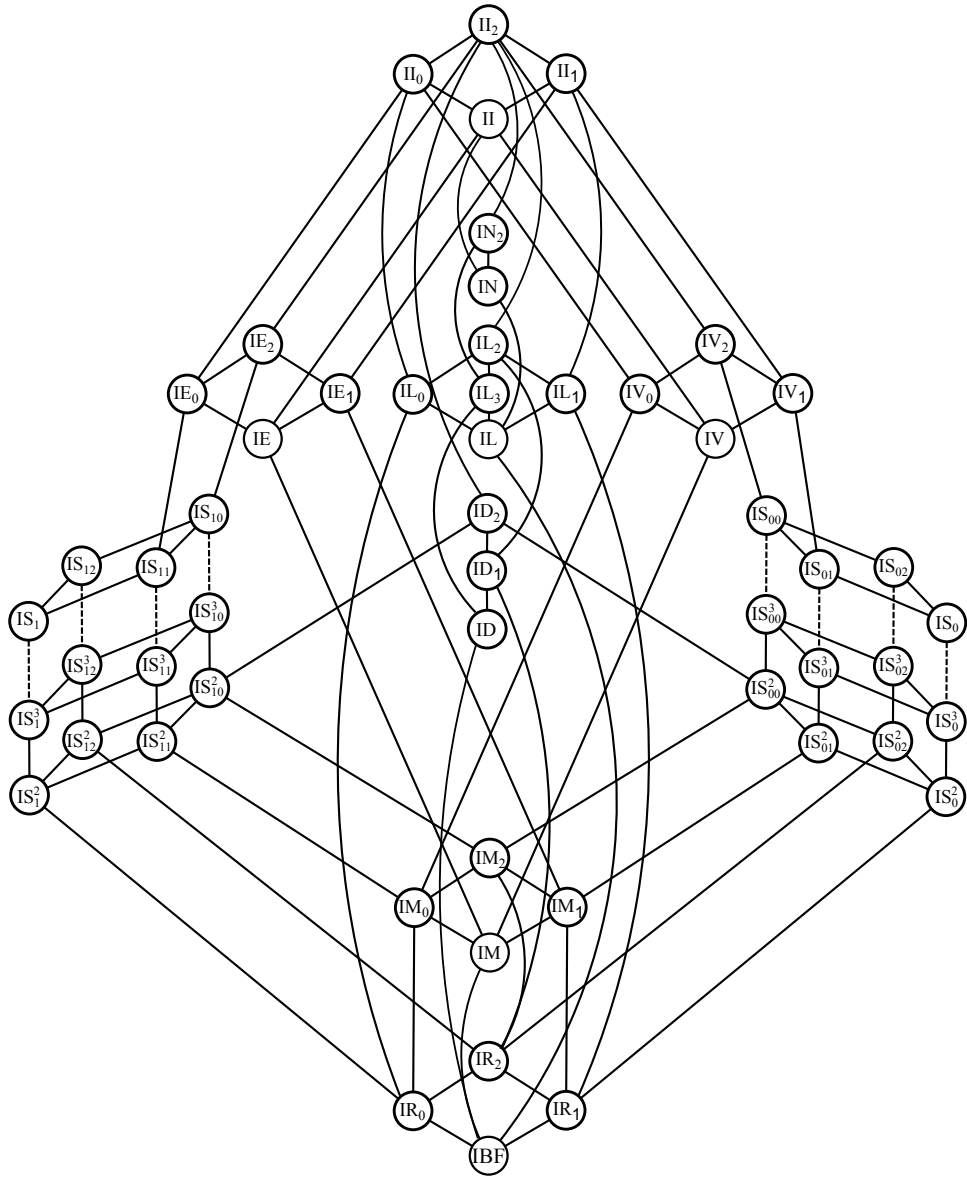


FIGURE 1
The lattice of Boolean co-clones

